Chapter 7 – Characteristic Excitation Response

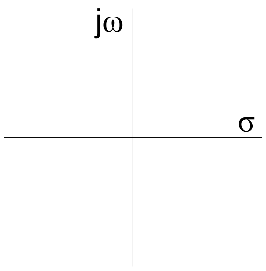
12.1 Introduction

The general structure of an equation or relationship follows a consistent structure. This may be written in any domain, but here is represented in the Laplace or s-domain.

$s = \sigma + j\omega$

$$E(s) = R(s) \times \frac{C(s)}{N(s)}$$

E is the excitation or energy source to the network.
R is the response of the network.
C is the characteristic or how the network is configured.
N is the network scaling
T is the reciprocal of C and *N*. *T* is called the transfer function.



To determine the roots of the characteristic function, there is no excitation influence. Therefore, the characteristic equation is set to zero. Then the roots can be determined without any outside influence.

The solution or response of the equation is simply the excitation operating on the transfer function.

$$R(s) = E(s) \times T(s)$$

$$T(s) = \frac{N(s)}{C(s)}$$

The beauty of this arrangement allows superposition of excitation functions. The response is simply the sum of each excitation multiplied by the transfer function.

$$R(s) = E_0(s)T(s) + E_f(s)T(s)$$

12.2 General approach

The general approach to calculating a system response is developed in seven steps.

	7-Steps
1	Network equation
2	Partial fraction expansion
3	Characteristic equation
4	Numerator
5	Partial fraction results
6	Inverse Laplace
7	Clean-up

12.2.1 General - network equation

The response is the product of the excitation and transfer function. For a physical system the most complex arrangement is a second order. The order depends on the number of storage elements. Real systems have two storage elements and always have an element that represents losses in the conversion between the two storage elements. The most complex numerator would correspond to the quadratic in the characteristic denominator.

$$R(s) = E(s)T(s)$$

$$= E(s)\frac{\left(a_2 s^2 + a_1 s + a_0\right)}{s^2 + as + b}$$

The excitation has a numerator, generally a magnitude or gain, and a denominator in powers of s. The denominator is called poles and the numerator is called zeros.

$$E(s) = \frac{En}{Ed}$$

12.2.2 General - characteristic

In a complete physical network, the characteristic function is represented by a second order. The order indicates the number of storage elements.

A second order system can arise under three conditions.

1. There can be two different elements of the same type. This will result in two unique roots.
   $$s^2 + as + b = (s+\alpha)(s+\beta)$$
2. There can be two identical elements of the same type. This will result in repeated roots. One of the solutions will have an extra time.
   $$s^2 + as + b = (s+\alpha)^2$$
3. There can be two different elements of different types. This will result in complex conjugate roots. The response will be sinusoidal.
   $$s^2 + as + b = (s+\alpha + j\omega)(s+\alpha - j\omega)$$

The most complex is the two different types of storage elements. Therefore, that assumption will be used as an illustration of a generic system analysis.

In the electrical case the two storage elements are the electric capacitor and the magnetic inductor.

The characteristic is a quadratic function. The quadratic is a complex relationship in conjugate pairs. To determine the roots of the characteristic, set the equation to zero. The roots of the characteristic equation are called poles.

Realizing the form, the quadratic term can often be factored. This is the preferred method.

$$s^2 + as + b = (s+\alpha + j\omega)(s+\alpha - j\omega)$$

Another approach is completing the square. In general a quadratic term can be expanded into a sum containing roots. The imaginary component may be zero, when one of the other forms arise.

$$\begin{aligned} s^2 + as + b &= s^2 + 2\alpha s + \alpha^2 + \omega^2 \\ &= (s+\alpha)^2 + \omega^2 \end{aligned}$$

The roots of the characteristic quadratic are expressed in terms of alpha and omega, the beginning and the ending of the Greek alphabet. Functionally, the frequency of oscillation or vibration is omega, ω. The stability is represented by alpha, α. Alpha is inversely related to the time it takes for the network to become stable.

The values of alpha and omega are found by equating the coefficients of the network to the expanded coefficients in terms of roots. Segregate the powers of s.

$$1s^2 = 1s^2$$

$$as = 2\alpha s$$

$$b = \alpha^2 + \omega^2$$

Calculate in terms of the desired root factor.

$$\alpha = \frac{a}{2}$$
$$\omega = \sqrt{b - \alpha^2}$$

This has assumed a quadratic form. A common alternate form does not involve frequency.

$$s^2 + as + b = (s+\alpha)(s+\beta)$$

There are a couple of variations to the structure of the network. The roots may be identical, resulting in a power. Alternately, only one root may exist. These are much simpler to resolve. But the techniques are the same.

If the characteristic is larger than a second order, it is because of multiplication of network functions or functions and excitation. These combined expressions could obviously be factored back to the original networks which will be a maximum of second order. Therefore, by developing a network with the most complex system, any other network can be analyzed by using superposition.

12.2.3 General - partial fraction expansion

$f(t)$	$\leftrightarrow$	$F(s)$
1	$\leftrightarrow$	$\frac{1}{s}$
t	$\leftrightarrow$	$\frac{1}{s^2}$
$e^{-\alpha t}$	$\leftrightarrow$	$\frac{1}{s+\alpha}$
$te^{-\alpha t}$	$\leftrightarrow$	$\frac{1}{(s+\alpha)^2}$
$\sin \omega t$	$\leftrightarrow$	$\frac{\omega}{s^2+\omega^2}$
$\cos \omega t$	$\leftrightarrow$	$\frac{s}{s^2+\omega^2}$
$e^{-\alpha t}\sin \omega t$	$\leftrightarrow$	$\frac{\omega}{(s+\alpha)^2+\omega^2}$
$e^{-\alpha t}\cos \omega t$	$\leftrightarrow$	$\frac{s+\alpha}{(s+\alpha)^2+\omega^2}$

The network correlation is expanded into a sum using partial fraction expansion. The expansion must be rearranged to get a form that is recognizable for Laplace table look-up.

$$R(s) = k_f\left[\frac{1}{Ed}\right] + \left[k_0 s + k_1\right]\left[\frac{1}{(s+\alpha)^2+\omega^2}\right]$$
$$R(s) = \mathit{forced} + \mathit{natural}$$

The first term is the forced response while the second term is the natural response. Since the denominator is frequency dependent, the numerator coefficient will also be in a frequency relationship.

$$k_0 s + k_1 = k_0(s+\alpha) + k_2\omega$$
$$k_1 = k_0\alpha + k_2\omega$$
$$k_2 = \frac{1}{\omega}\left(k_1 - k_0\alpha\right)$$

Now the response equation is in a Laplace recognizable form.

$$R(s) = k_f\left[\frac{1}{Ed}\right] + \left[k_0\left(s+\alpha\right) + k_2\omega\right]\left[\frac{1}{(s+\alpha)^2+\omega^2}\right]$$
$$= k_f\left[\frac{1}{Ed}\right] + \frac{k_0\left(s+\alpha\right)}{(s+\alpha)^2+\omega^2} + \frac{k_2\omega}{(s+\alpha)^2+\omega^2}$$

12.2.4 General - numerator coefficients

The value of the coefficients, k, on the right side can be found. Juxtapose the network correlation to the partial fraction expansion.

$$E(s)\frac{\left[a_2 s^2 + a_1 s + a_0\right]}{s^2+as+b} = k_f\left[\frac{1}{Ed}\right] + \frac{k_0\left(s+\alpha\right)}{(s+\alpha)^2+\omega^2} + \frac{k_2\omega}{(s+\alpha)^2+\omega^2}$$

Only the numerator is required for the coefficients. Multiply both sides by the denominator of the transfer function and any excitation.

$$En\left(a_2 s^2 + a_1 s + a_0\right) = k_f\left[s^2 + as + b\right] + k_0\left[Ed\right]\left[s+\alpha\right] + k_2\left[Ed\right]\omega$$

The coefficients on the right must equal the numerical coefficients on the left.

12.2.5 General - partial fraction results

Once the values of *k* are found, the response equation is expressed only with s and numbers. The only letters will be s.

$$R(s) = k_f\left[Ed\right] + \frac{k_0\left(s+\alpha\right)}{(s+\alpha)^2+\omega^2} + \frac{k_2\omega}{(s+\alpha)^2+\omega^2}$$

12.2.6 General - inverse Laplace

The inverse Laplace can be found directly from table look-up of each term.

$$r(t) = k_f\, f(t) + e^{-\alpha t}\left(k_0\cos\omega t + k_2\sin\omega t\right)$$

12.2.7 General - clean up: trig simplification

Using trigonometry, cosine and sine terms can be combined to obtain a different form.

$$k_0\cos\omega t + k_2\sin\omega t = k_i\cos\left(\omega t + \theta\right)$$

$$k_i = \sqrt{k_0^{\,2} + k_2^{\,2}}$$

$$\theta = \tan^{-1}\frac{k_2}{k_0}$$

Therefore, the time domain function has three components: an exponential decay coefficient, a cosine relationship, and an angular offset. This is the familiar general solution for second order systems.

$$r(t) = k_f\, f(t) + k_i e^{-\alpha t}\cos\left(\omega t + \theta\right)$$

EXAMPLES

Ex 1.3-1

Given: R(s)

Find: Partial fraction form

Situation R(s)	What is Partial Fraction Form
$\frac{5}{s^2+s}$	$\frac{k_f}{s} + \frac{k_0}{s+\alpha}$
$\frac{5s}{s^2+25}$	$\frac{k_f s}{s^2+\omega^2}$
$\frac{5}{s^2+3s+2}$	$\frac{k_1}{s+\beta} + \frac{k_0}{s+\alpha}$
$\frac{5}{s^2+9s+25}$	$\frac{k_0\omega}{\left(s+\alpha\right)^2+\omega^2}$

	$\frac{5}{s^3+9s^2+25s}$	$\frac{k_f}{s}+\frac{k_0\omega}{(s+\alpha)^2+\omega^2}$	

12.3 Physical network in time

A complete physical network is represented by a second order function.

$$v_0 = Ri + L\frac{di}{dt} + \frac{1}{C}\int_0^t i(t)dt + v_c(0)$$

The form indicates that the forcing function, excitation, or supply is, v_0. The initial condition of the forcing function is $v_c(0)$. The unknown response variable is *i*. The RLC are the network components, collective called impedance or opposition.

The relationship would be difficult to solve using calculus in the time domain. Therefore the Laplace is often preferred.

12.3.1 Transform it

The Laplace is taken for each of the terms in the equation.

$$Ri \leftrightarrow RI(s)$$

$$L\frac{di}{dt} \leftrightarrow \left[sI(s) - i(0)\right]$$

$$\frac{1}{C}\int_0^t i(t)dt \leftrightarrow \frac{I(s)}{sC}$$

$$v_c(0) \leftrightarrow \frac{v_c(0)}{s}$$

The source changes with the forcing function or excitation. Therefore, the transform will use a placeholder, $V(s)$.

$$v_0 \leftrightarrow V(s)$$

Combining the terms gives a single relationship again. This relationship includes all initial conditions. This is as complex as the problem can be.

$$V(s) = RI(s) + L\left[sI(s) - i(0)\right] + \frac{I(s)}{sC} + \frac{v_c(0)}{s}$$

12.3.2 Unknown arrangement

To solve the equation rearrange the relationship in terms of the unknown response, *I*. The objective at this point is to obtain the characteristic, then the resulting transfer function as discussed above. Therefore, the source is set to zero.

$$V(s) = 0 = I(s) \times C(s)$$

Manipulate the physical system, so the characteristic can be separated.

$$I(s)\left[R+sL+\frac{1}{sC}\right]=Li(0)-\frac{v_c(0)}{s}$$

$$I(s)=\frac{Li(0)-\frac{v_c(0)}{s}}{\left[Ls+R+\frac{1}{Cs}\right]}$$

12.3.3 Excitation on characteristics

Split the response into the excitation and characteristic components.

$$I(s)=\left[Li(0)-\frac{v_c(0)}{s}\right]\frac{\frac{s}{L}}{s^2+\frac{R}{L}s+\frac{1}{LC}}$$

The general appearance of the response includes excitation and transfer.

$$I(s)=E(s)\frac{N}{C(s)}$$

Note that in the characteristic circumstance, the excitation or source is simply the initial conditions. By design, the external forcing function was removed.

With the characteristic and transfer defined, any forcing function can be used for the excitation.

12.4 By inspection

All the information on the right side is known. The interesting situation is that the equation can be written almost by inspection from the network information. It is unnecessary to go through the time domain process.

12.4.1 Excitation

The excitation in the numerator includes three factors.

1. The initial condition of the response exists through the inductor, *i(0)*.
2. The initial condition of the excitation exists across the capacitor, *v(0)*.
3. A scaling function, *s/L*, which represents integration or summing.

The characteristic / transfer function depends solely upon the network elements RLC.

When the initial conditions and the forcing function are zero, there is no signal imposed on the network. Then the denominator of the transfer function is set to zero. This is the characteristic equation. It provides the roots of the network. This is the natural response.

The initial condition causes a transient response, since it is short-lived.

Other excitation or external forcing function can be applied to the network. The forcing provides the final value to the response, since the excitation persists.

12.4.2 Storage conditions

For a second order system, two initial conditions are required. The initial conditions are usually for each storage element. However, the derivative of one initial condition can be used to determine a second initial condition.

The storage element capacitor stores electric energy as voltage. The storage element inductor stores magnetic energy as current. Therefore, the initial conditions will correspond to the storage energy.

The capacitor will be an open circuit at its final state. The inductor will be a short circuit in its final state. Note that initial conditions are simply the final state of a previous circumstance before the switch changes status.

A singular function is one that is discontinuous. A switch is the physical implementation of a singular function.

To determine initial condition of an element, assume it has been in that status a very long time. Then calculate the current through the inductor and the voltage across the capacitor.

These would be the Thevenin / Norton equivalents, but can be calculated using any circuit analysis. Most frequently, the equivalent resistance and equivalent reactance elements are determined. Then a simple voltage or current calculation is performed using Ohm's Law.

12.5 Network initial conditions

	7-Steps
1	Network equation
2	Partial fraction expansion
3	Characteristic equation
4	Numerator
5	Partial fraction results
6	Inverse Laplace
7	Clean-up

With an equation describing the response, the process begins to find the inverse Laplace which gives a time response. The response of the physical network is the product of excitation and transfer components.

$$R(s) = E(s)T(s)$$

12.5.1 Initial - network equation

The first step to obtaining a response is to establish the network equation in terms of the response. For the second order system described above, the response has the initial condition excitation operating on the natural, characteristics.

$$I(s) = \left[Li(0) - \frac{v_c(0)}{s} \right] \frac{\frac{s}{L}}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

Rearrange from a product to a ratio with the characteristic in the denominator.

$$I(s) = \frac{\frac{s}{L}\left[Li(0) - \frac{v_c(0)}{s} \right]}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

12.5.2 Initial - partial fraction expansion

Set the network into a sum using partial fraction expansion. The expansion must be arranged to get a form that is recognizable for Laplace table look-up.

$$I(s)=\frac{k_0\left(s+\alpha\right)}{(s+\alpha)^2+\omega^2}+\frac{k_2\omega}{(s+\alpha)^2+\omega^2}$$

In order to determine the roots in the denominator, and the coefficients in the numerator, juxtapose the network with the expansion.

$$\frac{\frac{s}{L}\left[Li(0)-\frac{v_c(0)}{s}\right]}{s^2+\frac{R}{L}s+\frac{1}{LC}}=\frac{k_0\left(s+\alpha\right)}{s^2+2\alpha s+\alpha^2+\omega^2}+\frac{k_2\omega}{s^2+2\alpha s+\alpha^2+\omega^2}$$

The quadratic is a complex conjugate in the most complex arrangement. Alternately, the quadratic may be able to be factored into real roots without frequency.

12.5.3 Natural - characteristic equation

From the network in juxtaposition with the partial fraction expansion, set up the characteristic equation in the denominator. The frequency and time roots can be calculated.

$$s^2+\frac{R}{L}s+\frac{1}{LC}=s^2+2\alpha s+\alpha^2+\omega^2$$
$$=\left(s+\alpha\right)^2+\omega^2$$

The characteristic equation provides the roots of the network natural response. The values of alpha and omega are found by equating the coefficients of the network to the coefficients in terms of roots. Use the powers of s.

$$1s^2=1s^2$$
$$\frac{R}{L}s=2\alpha s$$
$$\frac{1}{LC}=\alpha^2+\omega^2$$

Calculate in terms of the desired root factor.

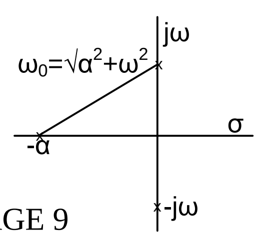
$$\alpha=\frac{R}{2L}$$
$$\omega=\sqrt{\frac{1}{LC}-\left(\frac{R}{2L}\right)^2}$$

The natural frequency of oscillation, ω_0, depends only on the storage elements, L and C.

$$\omega_0^{\ 2}=\frac{1}{LC}=\alpha^2+\omega^2$$

$$\omega_0^{\ 2}=\frac{1}{LC}=\alpha^2+\omega^2$$

The energy transfer between the storage devices causes oscillation or vibration. The inertial energy is from the magnetic inductor, L. The potential energy is from the electric capacitor, C. The natural frequency of oscillation, ω_0, results from this energy transfer.



The work and loss due to conversion is from the damper or resistance, *R*. This creates a decay frequency, α. Alpha is the reciprocal of the time constant.

The damped frequency, ω, is the result of the natural frequency minus the decay.

12.5.4 Initial - numerator coefficients

The numerator contains the initial conditions. The initial conditions exist from some incident prior to time zero. The initial magnetic current is stored in the inductor, while the initial electric voltage is stored in the capacitor.

For initial conditions, there is no forcing function. Rather the excitation is from the stored energy.

Reconsider the equation relating the network to the partial fraction expansion. Multiply both sides by the denominator of the transfer function and any excitation. Equate the numerators to provide the basis for the coefficients.

$$\frac{s}{L}\left[Li(0)-\frac{v_c(0)}{s}\right]=k_0\left(s+\alpha\right)+k_2\omega$$

The coefficients can be found by segregating in powers of s.

$$i(0)s-\frac{v_c(0)}{L}=k_0s+\left(k_0\alpha+k_2\omega\right)$$

Equate the powers of s.

$$i(0)s=k_0s$$

$$-\frac{v_c(0)}{L}=k_0\alpha+k_2\omega$$

Calculate the coefficients.

$$k_0=i(0)$$

$$k_2=-\frac{1}{\omega}\left[\frac{v_c(0)}{L}+\alpha i(0)\right]$$

12.5.5 Initial - partial fraction results

Now place the numerical value of the roots and the coefficients into the partial fraction expansion of the response.

$$I(s)=\frac{k_0\left(s+\alpha\right)}{(s+\alpha)^2+\omega^2}+\frac{k_2\omega}{(s+\alpha)^2+\omega^2}$$

In many cases, the standard form is obtained with a listing of the coefficients and roots without actually writing the roots into the equation. As noted below, the filled out form becomes very tedious and actually loses some of the understanding. On the other hand, it illustrates the interaction of the excitation on the characteristics.

$$I(s)=i(0)\frac{s+\dfrac{R}{2L}}{(s+\alpha)^2+\omega^2}-\frac{1}{\omega}\left[\frac{v_c(0)}{L}+\alpha i(0)\right]\frac{\sqrt{\dfrac{1}{LC}-\left(\dfrac{R}{2L}\right)^2}}{(s+\alpha)^2+\omega^2}$$

12.5.6 Initial - inverse Laplace

The objective was to find the response, *i(t)*, of the integral differential equation. The Laplace response can be converted to the time domain. Each term can be converted to the inverse Laplace directly from table look-up.

The general form of transient response is noted.

$$i(t) = e^{-at}\left[k_0 \cos \omega t + k_2 \sin \omega t\right]$$

Insert the characteristic poles and the numerator coefficients into the relationship.

This is the natural response to the initial conditions. It is a transient since the effect of the conditions vanish with time.

12.5.7 Initial - clean up

After the inverse Laplace is found, often combinations of terms in a different form are developed. These forms may be to match a particular system or to meet a defined structure.

Using trigonometry, cosine and sine terms can be combined to obtain a configuration using only one of them.

$$k_0 \cos \omega t + k_2 \sin \omega t = k_i \cos\left(\omega t + \theta\right)$$

$$k_i = \sqrt{k_0{}^2 + k_2{}^2}$$

$$\theta = \tan^{-1} \frac{k_2}{k_0}$$

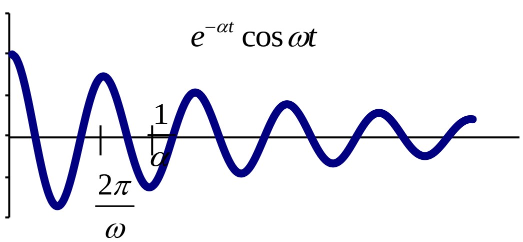
The time domain response solution to the initial conditions is a simple form of exponential decay operating on a shifted sinusoid.

$$i(t) = k_i e^{-\alpha t} \cos\left(\omega t + \theta\right)$$

The coefficients are the initial condition and combine into one initial term, k_i.

$$k_0 = i_0(0)$$

$$k_2 = -\frac{1}{\omega}\left(\frac{v_c}{L} + \alpha i_0(0)\right)$$



Dissect the coefficients into alternative forms.

$$k_2 = -\frac{1}{\omega L}\left[v_c(0) + \frac{R}{2} i(0)\right]$$

Note that the product of frequency and inductance defines reactance.

$$X_L = \omega L$$

The frequencies are from the characteristic equation.

$$\alpha = \frac{R}{2L}$$

$$\omega^2 = \frac{1}{LC} - \left(\frac{R}{2L}\right)^2$$

The natural frequency occurs from the oscillation between the storage elements.

$$\omega_0^{\ 2} = \alpha^2 + \omega^2 = \frac{1}{LC}$$

In summary, the time domain function has an exponential decay coefficient on a cosine relationship with an angular offset.

This is expected because of the limited time duration of the initial values. The cosine function is because of the oscillation from the storage components.

$$i(t) = k_i e^{-\alpha t} \cos\left(\omega t + \theta\right)$$

	EXAMPLES
Ex 1.3-1	Given: *R=12, L=2 mHy, C=20 pFd* Write transfer function when connected in series. $R + sL + \frac{1}{sC}$ $12 + 2\times10^{-3} s + \frac{1}{\left(20\times10^{-6}\right)s}$
Ex 1.3-2	Put transfer function in standard form $\frac{\frac{s}{L}}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$ $\frac{\frac{s}{2\times10^{-3}}}{s^2 + \frac{12}{2\times10^{-3}}s + \frac{1}{\left(2\times10^{-3}\right)\left(20\times10^{-6}\right)}}$
Ex 1.3-3	Write characteristic function $s^2 + \frac{R}{L}s + \frac{1}{LC}$ $s^2 + \frac{12}{2\times10^{-3}}s + \frac{1}{\left(2\times10^{-3}\right)\left(20\times10^{-6}\right)}$ $s^2 + 6\times10^3 s + 25\times10^6$
Ex 1.3-4	Find natural frequency $\omega_0 = \sqrt{\frac{1}{LC}}$ $= \sqrt{\frac{1}{\left(2\times10^{-3}\right)\left(20\times10^{-6}\right)}}$ $= \sqrt{25\times10^6}$ $= 5kHz$
	Find alpha

$$\alpha = \frac{R}{2L}$$
$$= \frac{12}{2\left(2\times10^{-3}\right)}$$
$$= 3\times10^{3}$$

Find damped frequency

$$\omega = \sqrt{\omega_0^{\ 2} - \alpha^2}$$
$$= \sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2}$$
$$= \sqrt{\left(5\times10^3\right)^2 - \left(3\times10^3\right)^2}$$
$$= 4\times10^3$$

Given initial excitation conditions.
Find initial response.

$$R(s) = E(s)T(s)$$
$$R(s) = \left[Li(0) - \frac{v_c(0)}{s}\right]T(s)$$
$$r(t) = k_i e^{-\alpha t}\cos\left(\omega t + \theta\right)$$

Find final response

$$R(s) = E(s)T(s)$$
$$R(s) = [0]T(s) = 0$$
$$r(t) = 0$$

The final value of a temporary condition is always zero, because of the exponential decay.

12.6 Forcing Functions

The forcing function signal provides the stimulus for a network. Since it is an on-going signal, the forcing function determines the steady state response.

$$R(s) = E(s)T(s)$$
$$= \frac{En}{Ed}\frac{\left(a_2 s^2 + a_1 s + a_0\right)}{s^2 + as + b}$$

The excitation takes on many forms. The most common ones are noted.

Time	Laplace	Type
$e(t) = 0$	$E(s) = 0$	No function, transient only
$e(t) = e$	$E(s) = \frac{E}{s}$	DC, step, constant input
$e(t) = e\sin\omega t$	$E(s) = E\frac{\omega}{s^2+\omega^2}$	AC, sinusoidal input

	7-Steps
1	Network equation
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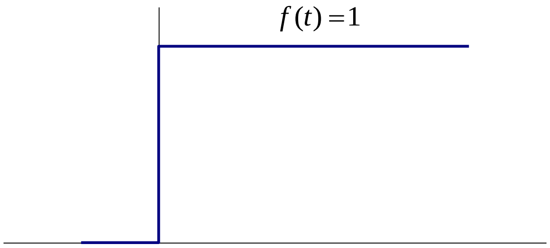
These values are multiplied by the transfer function of the network. Then they must be expanded by partial fraction expansion to obtain a form recognizable for a table look-up of the inverse Laplace.

12.7 Step excitation



The simplest excitation signal is also very common. The step is a simple direct current source that is turned on and keeps the same value during the entire operation.

$$V(s)=\frac{V}{s}$$

12.7.1 Step - network equation

Substitution of the excitation into the model response equation, gives an interesting result.

$$R(s)=\frac{V}{s}T(s)$$

The model is described specifically for a response current with a transfer function of a complete characteristic network. Write the network with the forcing function

$$I(s)=\left[\frac{V}{s}\right]\frac{\frac{s}{L}}{s^2+\frac{R}{L}s+\frac{1}{LC}}$$

12.7.2 Step - partial fraction expansion

Set the network into a sum using partial fraction expansion. The expansion must be arranged to get a form that is recognizable for Laplace table look-up.

$$I(s)=\frac{k_f}{s}+\frac{k_0\left(s+\alpha\right)}{(s+\alpha)^2+\omega^2}+\frac{k_2\omega}{(s+\alpha)^2+\omega^2}$$

In order to determine the roots in the denominator, and the coefficients in the numerator, juxtapose the network with the expansion.

$$\left[\frac{V}{s}\right]\frac{\frac{s}{L}}{s^2+\frac{R}{L}s+\frac{1}{LC}}=\frac{k_f}{s}+\frac{k_0\left(s+\alpha\right)}{s^2+2\alpha s+\alpha^2+\omega^2}+\frac{k_2\omega}{s^2+2\alpha s+\alpha^2+\omega^2}$$

Only one term is different from any other network with a forcing function. That term is the forcing function added to the natural characteristic response.

12.7.3 Step - characteristic roots

To determine the characteristic roots, the excitation is set to zero. Therefore, the roots of the characteristic equation are identical, regardless of the forcing function. The characteristic process is identical to the development in the natural, independent condition.

This process determines the frequencies and the poles.

α

ω

12.7.4 Step - numerator coefficients

Reconsider the equation relating the network to the partial fraction expansion. The objective is to find the value of the coefficients in terms of the network parameters. A combined technique works easiest.

For k_f, use the residue method. This is an iterative process.

1. Multiply both sides of the equation by one of the roots in the denominator.
2. Set s to equal that root.
3. Evaluate the remaining coefficient.

Repeat the process for each denominator root until all coefficients are found.

Because of the structure of this particular problem, using the residue to find k_f is unnecessary, but will be included as an illustration.

12.7.4.1 Residue

Multiply both sides of the equation by *s*.

$$\frac{V}{L}s = k_f + sk_0\left[s+\alpha\right] + sk_2\omega$$

Set s to zero, then evaluate the residue coefficients in the numerator.

$$\left.\frac{\frac{V}{L}s}{s^2+\frac{R}{L}s+\frac{1}{LC}}\right|_{s=0} = k_f + \frac{sk_0\left(s+\alpha\right)}{s^2+2\alpha s+\alpha^2+\omega^2} + \frac{sk_2\omega}{s^2+2\alpha s+\alpha^2+\omega^2}$$

In effect, only the forcing function coefficient remains.

$$0 = k_f$$

12.7.4.2 Algebra

Multiply both sides of the equation by the next root. The complex conjugate mathematics would be most tedious. Therefore, the algebra method of equating the coefficients is much more straightforward.

$$\left[\frac{V}{s}\right]\frac{\frac{s}{L}}{s^2+\frac{R}{L}s+\frac{1}{LC}} = \frac{k_f}{s} + \frac{k_0\left(s+\alpha\right)}{s^2+2\alpha s+\alpha^2+\omega^2} + \frac{k_2\omega}{s^2+2\alpha s+\alpha^2+\omega^2}$$

Multiply both sides of the equation by the denominator of the transfer function and any excitation. In this case use $s\left(s^2+\frac{R}{L}s+\frac{1}{LC}\right)$.

$$\frac{V}{L}s = k_f\left[s^2 + as + b\right] + k_0 s\left[s + \alpha\right] + k_2 s\omega$$

Evaluate by equating the numerators to provide the basis for the coefficients.

$$\frac{V}{L}s = k_f s^2 + k_f as + k_f b + k_0 s^2 + k_0\alpha s + k_2\omega s$$

The coefficients can be found by segregating in powers of s.

$$0 = \left(k_f + k_0\right)s^2$$

$$\frac{V}{L}s = (k_f a + k_0\alpha + k_2\omega)s$$

$$0 = k_f b$$

Interpret the equations into individual coefficients. Note that one, k_f, has already been determine but is included for illustration purposes.

$$k_f = 0$$

$$k_0 = -k_f = 0$$

$$k_2 = \frac{1}{\omega}\frac{V}{L}$$

12.7.5 Step - partial fraction results

Substitute the coefficients and the roots from the characteristic into the partial fraction expansion. This will provide the Laplace solution to the response.

$$I(s) = k_f\frac{1}{s} + k_0\frac{s+\alpha}{(s+\alpha)^2 + \omega^2} + k_2\frac{\omega}{(s+\alpha)^2 + \omega^2}$$

The actual values may be substituted.

$$I(s) = 0 + 0 + \frac{V}{\omega L}\frac{\omega}{(s+\alpha)^2 + \omega^2}$$

The response of the step is the step magnitude on an oscillating term.

$$I(s) = \frac{V}{\omega L}\left[\frac{\omega}{(s+\alpha)^2 + \omega^2}\right]$$

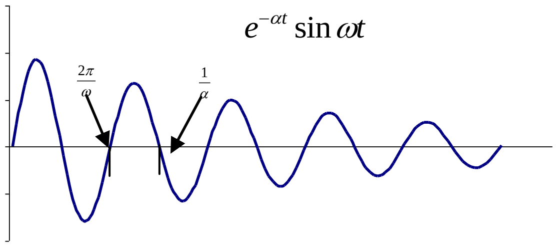
The denominator is the reactance.

$$X_L = \omega L$$

Usually, the values for the characteristics are not substituted, so the form is easier to see. Then the roots and coefficients are inserted.

12.7.6 Step - inverse Laplace

Using table look-up, find the inverse Laplace individually for each of the sums. Then add the terms to obtain the complete time domain relationship.



$$i(t) = \frac{V}{\omega L}\left[e^{-\alpha t}\sin\omega t\right]$$

This is a response to be expected. With the inductor and capacitor, there will be oscillation called ringing. The resistor causes losses and decay in the response. At the end of time, the decay will be to zero.

The physical components also indicate the response. The capacitor is an open circuit to direct current, or a step. It will eventually develop a charge that matches the potential from the supply.

12.7.7 Step - clean up

After the inverse Laplace is found, often combinations of terms in a different form are developed. These forms may be to match a particular system or to meet a defined structure.

For this problem, no reduction is available.

However, the following setup remains for analysis with different excitation. Using trigonometry, cosine and sine terms can be combined to obtain a configuration using only one of them.

$$k_0 \cos \omega t + k_2 \sin \omega t = k_i \cos\left(\omega t + \theta\right)$$

$$k_0 = i_0(0)$$

$$k_2 = -\frac{1}{\omega}\left(\frac{v_c}{L} + \alpha i_0(0)\right)$$

$$k_i = \frac{V}{L}\sqrt{1^2 + \left(\frac{\alpha}{\omega}\right)^2}$$

$$= \frac{V}{L}\sqrt{\frac{\omega^2 + \alpha^2}{\omega^2}} = \frac{V}{L}\frac{\omega_0}{\omega}$$

$$\theta = \tan^{-1}\frac{k_2}{k_0} = \tan^{-1}\frac{\alpha}{\omega}$$

The coefficients combine into one initial term, k_i, and the root frequency information. The frequencies are from the characteristic equation.

$$\alpha = \frac{R}{2L}$$

$$\omega^2 = \frac{1}{LC} - \left(\frac{R}{2L}\right)^2$$

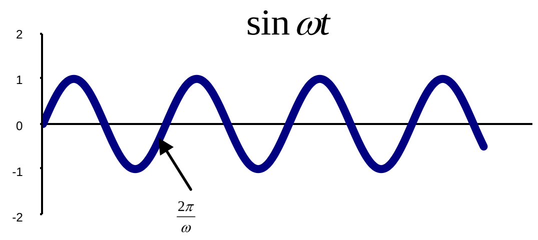
The natural frequency occurs from the oscillation between the storage elements.

$$\omega_0{}^2 = \alpha^2 + \omega^2 = \frac{1}{LC}$$

In summary, the time domain function has an exponential decay coefficient on a sine relationship with an angular offset. This is expected because of the damping. The sine function is because of the oscillation from the storage components.

12.8 Sinusoid excitation

The oscillating excitation signal is very common. The sinusoid is a simple alternating current source that is turned on and oscillates about the same value during the entire operation.



$$V(s) = V\frac{\omega}{s^2+\omega^2}$$

12.8.1 Sine - network equation

Substitution of the excitation into the model response equation, gives an interesting result.

$$R(s) = V\frac{\omega}{s^2+\omega^2}T(s)$$

	7-Steps
1	Network equation
2	Partial fraction expansion
3	Characteristic equation
4	Numerator
5	Partial fraction results
6	Inverse Laplace
7	Clean-up

The model is described specifically for a response current with a transfer function of a complete characteristic network. Write the network with the forcing function

$$I(s) = \left[V\frac{\omega}{s^2+\omega^2}\right]\frac{\frac{s}{L}}{s^2+\frac{R}{L}s+\frac{1}{LC}}$$

12.8.2 Sine - partial fraction expansion

Set the network into a sum using partial fraction expansion. The expansion must be arranged to get a form that is recognizable for Laplace table look-up.

$$I(s) = \frac{k_f}{s^2+\omega^2}+\frac{k_0\left(s+\alpha\right)}{(s+\alpha)^2+\omega^2}+\frac{k_2\omega}{(s+\alpha)^2+\omega^2}$$

In order to determine the roots in the denominator, and the coefficients in the numerator, juxtapose the network with the expansion.

$$\left[V\frac{\omega}{s^2+\omega^2}\right]\frac{\frac{s}{L}}{s^2+\frac{R}{L}s+\frac{1}{LC}} = \frac{k_f}{s^2+\omega^2}+\frac{k_0\left(s+\alpha\right)}{s^2+2\alpha s+\alpha^2+\omega^2}+\frac{k_2\omega}{s^2+2\alpha s+\alpha^2+\omega^2}$$

Only one term is different from any other network with a forcing function. That term is the forcing function added to the natural characteristic response.

12.8.3 Sine - characteristic roots

To determine the characteristic roots, the excitation is set to zero. Therefore, the roots of the characteristic equation are identical, regardless of the forcing function. The characteristic process is identical to the development in the natural, independent condition.

This process determines the frequencies and the poles.

$$\alpha$$

$$\omega$$

12.9 General response

Obviously, the transfer function determines the denominator roots. The excitation comes from both the initial conditions and the forcing function. As a result, these three values define the response equation.

The final value is the level about which the response oscillates.

The final response is from the forcing function excitation.

$$i(t) = k_f - k_f e^{-\alpha t}\cos(\omega t + \theta)$$

The initial condition response depends on the energy stored in the electric capacitor and magnetic inductor elements in time before zero.

$$i(t) = k_i e^{-\alpha t}\cos(\omega t + \theta)$$

Using the principal of superposition, the response due to all excitation can be summed.

$$i(t) = k_f + \left[k_i - k_f\right] e^{-\alpha t}\cos(\omega t + \theta)$$

This relationship is for a step input to the forcing function. Other waveforms will provide a similar result.

The general solution to the second order equation can be express in terms of the initial, *I*, and final, *F*, conditions.

$$f(t) = F + \left[I - F\right] e^{-\alpha t}\cos(\omega t + \theta)$$

$$i(t) = k_f + \left[k_i - k_f\right] e^{-\alpha t}\cos(\omega t + \theta)$$

The forcing function creates the final condition. In a sinusoid, this is the steady-state value. The sinusoidal component oscillates about this offset.

The initial condition is the excitation from activity before time zero.

The Laplace transform lets us develop this general solution in the time domain without ever using a derivative or integral calculus calculation.

The general relationship permits us to solve numerous problems without resorting to the detailed calculations we have just perused.

12.10 Situation awareness.

Equations are a model or mathematical representation of a physical phenomenon. They are a short-hand photograph of how items interact.

An equation has the following components.

1. Measurement variables
   a. Forcing or source function.
   b. Unknown flow rate.
   c. Time relationship
2. Characteristic network consisting of 3 elements – R, L, C.
3. Initial conditions at time zero.
   a. from forcing function.

b. on unknown flow rate.

The objective to solving an equation is to find the unknown flow rate for the network conditions. Since time is one of the measurement variables in physical systems, most equations are written in a form containing time and how the measurements change with time.

Laplace transforms do not exist in a physical sense. They are a representation of a time problem. The purpose of Laplace transforms is to convert a time-related problem from calculus form into a form that can be solved by algebra. Then the algebra form is converted back to time for the solution.

The steps to using Laplace transforms follow.

	7-Steps
1	Network equation
2	Partial fraction expansion
3	Characteristic equation
4	Numerator
5	Partial fraction results
6	Inverse Laplace
7	Clean-up

1. Network equation: Convert the time problem to a Laplace form by using table look-ups. Rearrange the equation so the unknown is on the left side. The right side will be a ratio.
2. Partial fraction expansion: Set the network into a sum using partial fraction expansion. The expansion must be arranged to get a form that is recognizable for Laplace table look-up.
3. Characteristic equation: The denominator is the characteristic equation. It is the network definition. By equating coefficients, find the roots.
   a. Alpha is the stability.
   b. Omega is the frequency
4. Numerator: Use partial fraction expansion to isolate the coefficients.
   a. The first coefficient, k_0, is stability scaling or magnitude.
   b. The second coefficient, k_2, is the frequency scaling or magnitude.
   c. The third coefficient, k_f, is the forcing function or final value.
5. Partial fraction results: Combine the partial fraction equation using the expanded numerator and characteristic equation coefficients.
6. Inverse Laplace: Convert the Laplace to time by using the inverse table look-up.
7. Clean-up: Rearrange the results using trigonometric or other tools to obtain a structure that fits the system.
8.

EXAMPLES

Ex 1.3-1

Given: *R=12, L=2 mHy, C=20 pFd*

Given: R=100, L=10, C=1x10^{-3}
V_0=5, $v_c(0)$=0, i(0)=0
Find I(s)

$V_0(s) = \frac{V}{s}$ This is a dc or step input.

$$I(s) = \frac{\left[i(0)s - \frac{v_c(0)}{L} \right] + \frac{s}{L}V(s)}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

	$$I(s)=\frac{\left[0s-{}^{0}\!/\!_{10}\right]+{}^{5}\!/\!_{10}}{s^2+\frac{100}{10}s+\frac{1}{10\times10^{-3}}}$$ $$I(s)=\frac{{}^{1}\!/\!_{2}}{s^2+10s+10^4}$$
Ex 1.3-2	Given: R=100, L=10, C=1x10^{-3} Find: α and ω $$CE=s^2+\frac{R}{L}s+\frac{1}{LC}$$ $$=(s+\alpha)^2+\omega^2$$ $$=s^2+2\alpha s+\alpha^2+\omega^2$$ $$\therefore\alpha=\frac{R}{2L}\quad\&\quad\omega^2={}^{1}\!/\!_{LC}-\alpha^2$$ $$\alpha=\frac{100}{20}=5\ \ \&\ \ \omega^2=10^4-5^2\approx10^4$$
Ex 1.3-3	Find: numerator coefficient $$k_3\omega=\frac{V_0}{L}$$ $$k_3=\frac{V_0}{\omega L}=\frac{5}{(10^2)10}=5\times10^{-3}$$
Ex 1.3-4	Find: I(s) in terms of α and ω. Ignore non-sinusoid components. $$I(s)=\frac{k_3\omega}{(s+\alpha)^2+\omega^2}$$ $$I(s)=5\times10^{-3}\frac{10^2}{(s+5)^2+(10^2)^2}$$
	Find: inverse Laplace to give i(t). $$I(s)=5\times10^{-3}e^{-5t}\sin100t$$

12.8 Example