Chapter 8 – Transients & RLC

8.1 Introduction

Signals and waveforms are not normally part of the study for electric machines. However, with the growing amount of electronic control and with distortion on the power line from switched mode power supplies, the waveform is often complex. Therefore, this chapter is provided as a reference to assist with those challenges.

Signals that are encountered can be a constant, direct current (DC), they can be repetitive, alternating current (AC), or they can be short term, transients. The circuit elements respond differently to each type signal. This chapter will address waveforms and tools to analyze their impact on systems performance. The time domain signal response or solution contains all the components.

- $y(t) = F + (I - F)e^{-t/\tau}\cos(\omega t + \theta)$

8.3 Transients

Transients are waveforms that exist for a short period of time. Waveforms are determined by the circuit elements. Since there are only three elements, the most complex circuit is a second order. The characteristic solution for a systems circuit is the time varying equation that describes the exponential decay after a signal is applied. The variable, *y*, can represent either current or voltage.

$$y(t) = F + (I - F)e^{-t/\tau}\cos(\omega t + \theta)$$

where

F = final value (t=∞)

I = Initial Value (t=0)

τ = time constant

$$\omega = 1/\sqrt{LC}$$

8.3.1 First Order Transients

First order systems are very common, since they are the model of a simple system. First order systems have a resistor and either a capacitor or an inductor.

First Order Circuits

RC or RL

Form:

$$v = L\frac{di}{dt} + Ri$$

$$i = C\frac{dv}{dt} + \frac{v}{R}$$

Characteristic Solution

Response to a step input (DC)

$$y(t) = F + (I - F)e^{-t/\tau}$$

For Capacitor: $i = C\dfrac{dv}{dt}$

Voltage does not change instantaneously

Open circuit under DC conditions

Capacitor discharges to $V_C(\infty) = 0$

Initial voltage = source voltage

For Inductor: $v = L\frac{di}{dt}$

Current does not change instantaneously

Short circuit under DC conditions

Inductor dissipates to $I_L(\infty) = I$

Initial current = source current

Process:

Find τ

$\tau =$ time constant=RC or L/R

= time for exponent to be $e^{-1} \Rightarrow \frac{1}{e}$

use equivalent circuit w/o source to get RC or RL (Thevenin Impedance)

deactivate all the sources and replace with internal Z

reduce to single equivalent RC or RL

Find $y(0)$

use circuit (KVL) w/ element as source

Find y(final)

use circuit (KVL) w/ element as limit

Plot:

Initial slope = $\frac{F-I}{\tau}$

Transfer function = $\frac{response}{excitation} = \frac{output}{input}$

8.3.2 RL Circuits

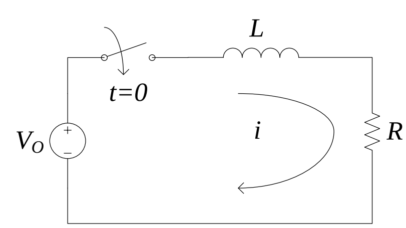
Standard calculus form

$$V_o = L\frac{di}{dt} + Ri$$

Inductor is short circuit in final state.

$$i(L)_0 = 0$$

$$i(L)_f = \frac{V_o}{R}$$



$$\tau = \frac{L}{R}$$

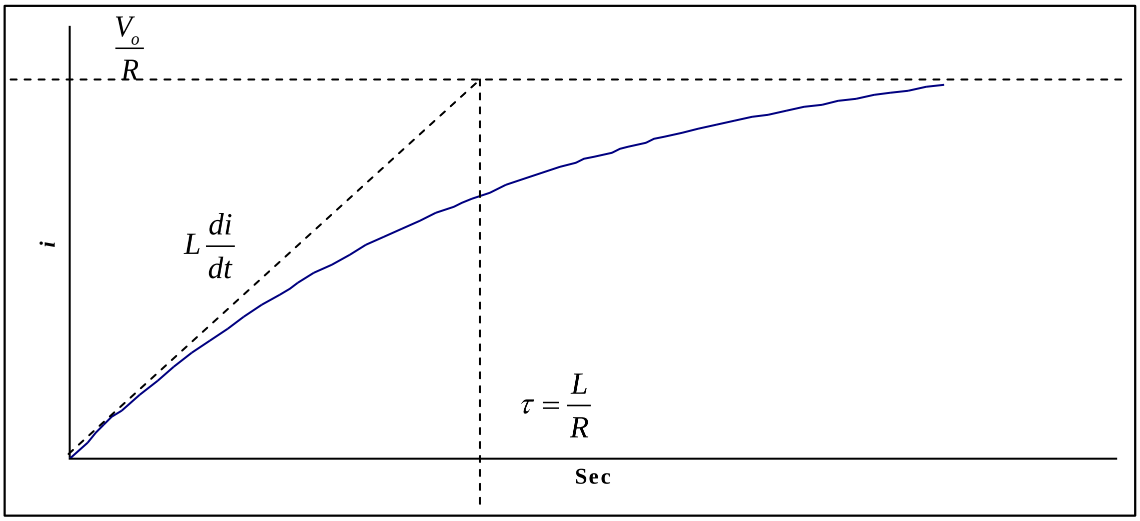
General solution

$$y(t) = F + (I - F)e^{-t/\tau}$$

Current solution

$$i = \frac{V_o}{R} + (0 - \frac{V_o}{R})e^{-\frac{t}{\tau}}$$

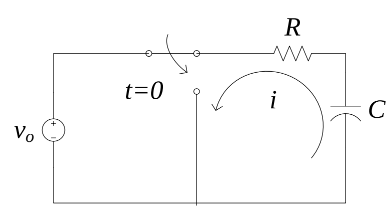
$$i = \frac{V_o}{R}(1 - e^{-\frac{Rt}{L}})$$



8.3.3 RC Circuits

Standard calculus form

$$i = C\frac{dv}{dt} + \frac{v}{R} \text{ (calculus form)}$$



Capacitor is open circuit in final state.

i, v_c cannot change instantaneously

$$v(C)_0 = V_o$$

$$v(C)_f = 0$$

$$\tau = RC$$

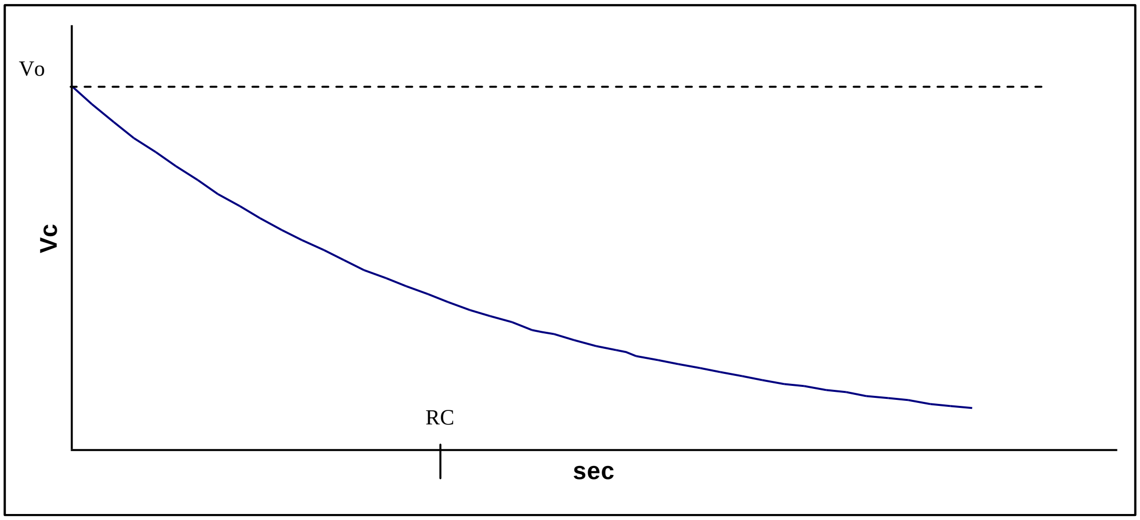
General solution

$$y(t) = F + (I - F)e^{-t/\tau}$$

Voltage solution

$$v = V_F + (V_I - V_F)e^{-t/\tau}$$

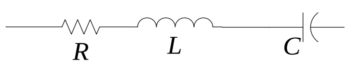
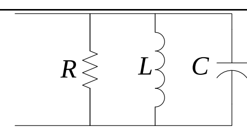
$$v_c = V_o e^{-t/\tau}$$



8.9 RLC System Response

8.9.1 RLC Equations

The three elements, RLC can be arranged in series or its dual parallel. This is a second order system. The analysis of the circuit can be made in many domains. Typically the time domain is the starting point. However, the Calculus required makes the mathematic interpretation difficult. For that reason numerous transforms are used. The math of the transforms will not be developed, but the correspondence is apparent from the table. The duality of the circuits is intriguing.

Function	**Series**	**Parallel**
Reference	Same current through all elements	Same voltage across all elements
Diagram	 	
Fundamental	$v(t) = L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q$	$i(t) = C\frac{d^2\phi}{dt^2} + \frac{1}{R}\frac{d\phi}{dt} + \frac{1}{L}\phi$
Time	$v(t) = L\frac{di}{dt} + Ri + \frac{1}{C}\int idt$	$i(t) = C\frac{dv}{dt} + Rv + \frac{1}{C}\int vdt$
LaPlace	$V(s) = (Ls + R + \frac{1}{Cs})I(s)$	$I(s) = (Cs + \frac{1}{R} + \frac{1}{Ls})V(s)$
Sinusoidal Steady State	$V(j\omega) = (j\omega L + R + \frac{1}{j\omega C})I(j\omega)$	$I(j\omega) = (j\omega C + \frac{1}{R} + \frac{1}{j\omega L})V(j\omega)$

Several observations can be made about the relationships.

$\frac{d}{dt} = s = j\omega$	$\int dt = \frac{1}{j\omega} = \frac{1}{s}$

$\frac{dq}{dt} = q' = i$	$\frac{d\phi}{dt} = \phi' = v$

8.9.2 System Response

The system response is the solution to the second order equation.

$$y(t) = F + (I - F)e^{-t/\tau}\cos(\omega t + \theta)$$

Time constant is the time it takes for a signal to settle so that the exponential decay.

$$\tau = RC = \frac{L}{R} = \text{time constant}$$

8.9.3 Characteristic Transfer

Transfer functions are often used as a model for a system.

Function	**Series**	**Parallel**
Transfer function	$Y(s) = \frac{I(s)}{V(s)}$	$X(s) = \frac{V(s)}{I(s)}$
Characteristic	$Y(s) = \frac{1}{Ls + R + \frac{1}{Cs}}$	$Z(s) = \frac{1}{Cs + \frac{1}{R} + \frac{1}{Ls}}$
Standard form	$Y(s) = \frac{s/L}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$	$Z(s) = \frac{s/C}{s^2 + \frac{s}{RC} + \frac{1}{LC}}$
Resonance	$Y(s) = \frac{s/L}{s^2 + \Delta\omega s + \omega_0^2}$	$Z(s) = \frac{s/C}{s^2 + \Delta\omega s + \omega_0^2}$

8.9.4 Resonance

Frequency is inversely related to time. Angular frequency is one complete revolution of cycle of the frequency.

$$\omega = 2\pi f$$

Resonance is a very significant concept that may be a boon or ban to electrical systems. Resonance is the frequency where the magnetic (or inductor) energy equals the electric (or capacitor) energy.

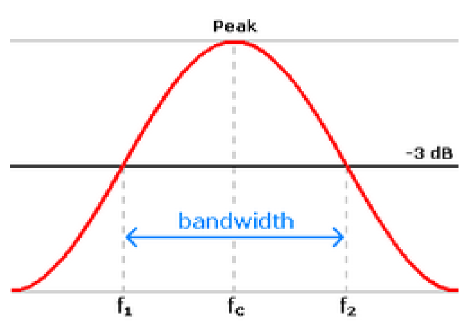
$$\omega_0 = \frac{1}{\sqrt{LC}}$$

Since the energies are balanced, it flows from one to the other resulting in a sinusoidal frequency. The natural frequency is the oscillation determined by the physical properties. Resonant frequency is a created oscillation that matches the natural frequency. Resonance is the frequency at which the input impedance is purely real or resistive.

The frequency response has a roll-off on either side. The transition is called the cut-off frequency.

$$\omega_0^2 = \omega_{cL}\omega_{cH}$$

Bandwidth, $\Delta\omega$, is the range between the upper and lower cut-off frequencies. The bandwidth is also called the pass band or bandpass.



$$\Delta\omega = \omega_{cH} - \omega_{cL}$$

$$\omega_{cL} = \omega_0 - \frac{\Delta\omega}{2}$$

$$\omega_{cH} = \omega_0 + \frac{\Delta\omega}{2}$$

Quality factor or selectivity is the sharpness of the peak at resonance.

$$Q = \frac{\omega_0}{\Delta\omega}$$

Damping is the effect of resistance on the rate that a signal is stabilized to steady state. Undamped implies that there is no resistance, R=0. The *damping coefficient* is dependent on the natural frequency and is inversely proportional to twice the quality factor. Some authors use the symbol alpha, α, rather than zeta, ς. Note this is also the real term of the LaPlace, σ.

$$\zeta = \frac{\omega_0}{2Q} = \frac{R}{2\sqrt{L/C}} = \frac{\text{actual damping}}{\text{critical damping}}$$

The range of values for the camping coefficient reflects how quickly the waveform will settle and whether it will overshoot. Under-damping results in oscillations or ringing, over-damping results in a slow exponential approach to stability, critical-damping is the transition between oscillations and exponential.

$$\begin{aligned} \zeta < 1 &\rightarrow \text{ under-damped = oscillation} \\ \zeta = 1 &\rightarrow \text{ critical-damped = transition} \\ \zeta > 1 &\rightarrow \text{ overdamped = exponential} \end{aligned}$$

The relationship between the various factors can be described in terms of the quality factor.

$$Q = \frac{\omega_0}{\Delta\omega} = \frac{\omega_0}{2\zeta}$$

Damped resonance, ω_d, is a shift from the resonant frequency caused by the damping.

$$\omega_d^2 = \omega_0^2 - \zeta^2$$

The root of the characteristic equation has the real part as damping coefficient and the imaginary part as the damped resonance. For the second order, there are two roots.

$$s_{1,2} = -\zeta \pm j\omega_d$$

8.9.5 Series Parallel Duality

Comparison of the standard form and the resonance equation reveal the duality of impedance and admittance. The symmetry of the duality resolves to a reciprocal form at resonance.

Function	**Series**	**Parallel**
Quality factor	$Q = \frac{X}{R}$	$Q = \frac{R}{X}$
Quality factor	$Q = \frac{1}{R}\sqrt{\frac{L}{C}}$	$Q = R\sqrt{\frac{C}{L}}$

8.9.6 First Order

A first order system has a resistor and either a capacitor or inductor. Therefore, there is no oscillation. However, there is still a cut-off frequency that is the inverse of the time constant.

$$R(j\omega) + \frac{1}{C} = 0 \Rightarrow \omega_c = \frac{1}{RC}$$ Time Constant = RC

$$L(j\omega)^2 + R(j\omega) = 0 \Rightarrow \omega_c = \frac{R}{L}$$ Time Constant = $\frac{L}{R}$

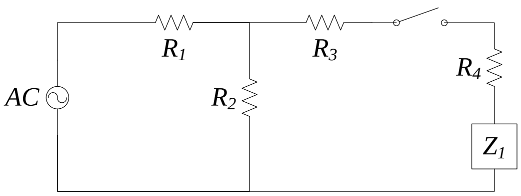
End of chapter

8.10 Exemplars

An exemplar is typical or representative of a system. These examples are representative of real world situations.

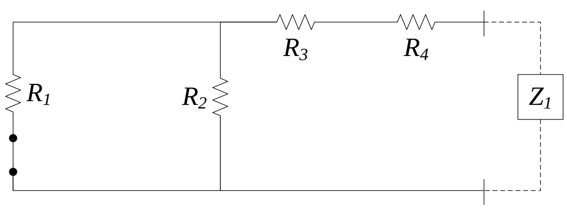
Problem 1

Consider the circuit shown below. R_1 and R_2 are 5Ω resistors. R_3 is a 10Ω resistor and R_4 is a 15Ω resistor. Z_1 is a 20μF capacitor, and V_1 is a 120V source. The time constant of the circuit is most nearly



(A) 85 μS
(B) 138 μS
(C) 550 μS
(D) 400 μS

SOLUTION:
Redraw the circuit to make it easier to see



The resistances can be combined to determine the equivalent resistance of the circuit.

$$R_{eq} = R_4 + R_3 + (R_1 // R_2)$$
$$R_{eq} = 15\Omega + 10\Omega + (5\Omega // 5\Omega)$$
$$R_{eq} = 25\Omega + 2.5\Omega = 27.5\Omega$$

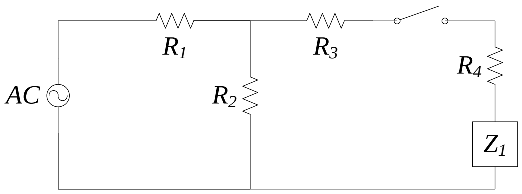
The time constant of a RC circuit is

$$\tau = R_{eq}C_{eq} = (27.5\Omega)(20\mu F) = 550\mu S$$

The answer is (C)

Problem 2

Consider the circuit shown in the problem above, and recreated below. R1 and R2 are 15Ω resistors. R3 is a 20Ω resistor and R4 is a 15Ω resistor. Z1 is a 20mH capacitor, and V1 is a 120V, 60Hz source. The switch has been closed for a significant period of time. The voltage across the inductor is most nearly.



(A) 25.4∠55°

(B) 10.5∠80°

(C) 50.7∠-60°

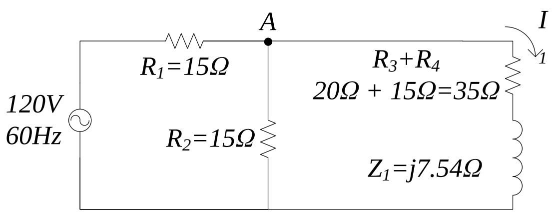
(D) 61.8∠90°

SOLUTION

Impedance of the Inductor Z_1

$$Z_1 = j2\pi 60(20mH) = j7.54\Omega$$

Redraw with all impedances



$$R_A = R_3 + R_4 + Z_1 = 35 + j7.54\Omega$$

$$R_B = 15\Omega // R_A = 10.6 + j0.664\Omega$$

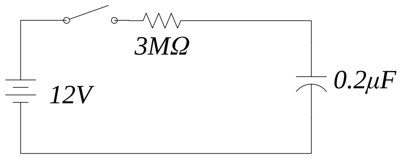
$$V_A = \frac{120(R_B)}{(R_1 + R_B)} = \frac{120V(10.6 + j0.664\Omega)}{(25.6 + j0.664\Omega)} = 49.73 + j1.822$$

$$V_{Z_1} = \frac{V_A(Z_1)}{(R_A + Z_1)} = \frac{(49.73 + j1.822V)(j7.54\Omega)}{(35 + j7.54\Omega)} = 1.83 + j10.31V = 10.5\angle 79.94V$$

The answer is (B)

Problem 3

What is the time constant of the figure shown?



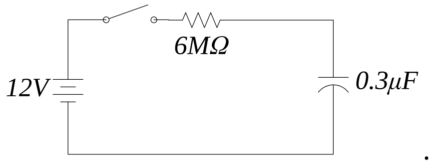
SOLUTION:

The time constant of an RC circuit is

$$\begin{aligned}\tau &= RC \\ &= \left(3\times10^{6}\right)\left(0.2\times10^{-6}\right) \\ &= 0.6 \text{ seconds}\end{aligned}$$

Problem 4

In the figure below, the switch has been open for a significant period of time and is closed at *t*=0. What is the current in the capacitor at $t=0_+$?



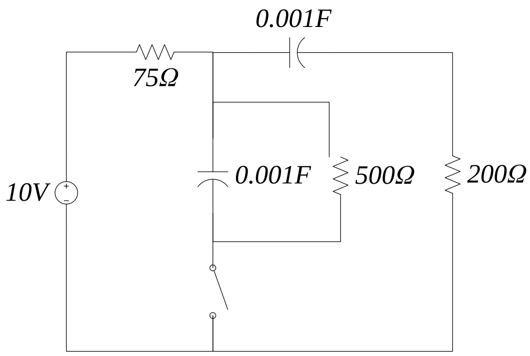
SOLUTION:

The capacitor, at t=0+, acts as a short circuit. The current through the capacitor then is determined by the voltage and the resistance

$$i_c\Big|_{t=0+} = \frac{V}{Z} = \frac{12V}{6\times10^6\Omega} = 2\times10^{-6}A$$

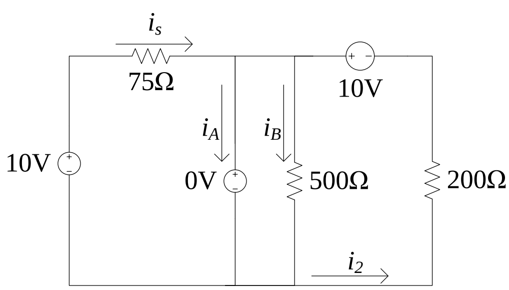
Problem 5

In the figure below, the switch has been open for a significant period of time, and is then closed at t=0. What is the current through the two capacitors at $t=0_+$?



SOLUTION:

If the switch is opened for a significant period of time the capacitor on top of the circuit is charged to 10V, and the capacitor in the middle of the circuit is discharged to 0V. At $t=0_+$, the capacitors are modeled as voltage sources with the charged voltages. The equivalent circuit is shown below



The voltage across the 500Ω resistor is 0V, so i_B=0A.

KVL on the left loop is

$$10V - i_s(75\Omega) - 0V = 0$$
$$\Rightarrow i_s = \frac{10V}{75\Omega} = 0.1\overline{33}A$$

KVL on the right loop is

$$10V + 0V + i_2(200\Omega) = 0$$
$$\Rightarrow i_2 = \frac{10V}{200\Omega} = 0.05A$$

KCL

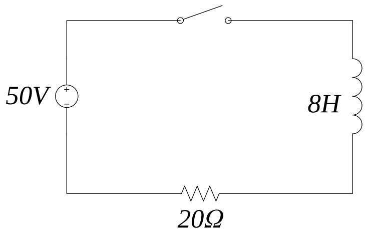
$$i_s - i_A - i_B - i_2 = 0$$
$$0.133A - i_A - 0 - 0.05A = 0$$
$$i_A = 0.133 - 0.05 = 0.083A$$

The current through the top capacitor is i_2=0.05A

The current through the middle capacitor is i_A = 0.083A

Problem 6

In the figure below, the switch has been open for a significant period of time. The switch is closed at t=0. Find the current through the resistor at $t=0_+$, and at t=1.25s. Find the energy in the inductor at t=1.25s.



SOLUTION:

The current in an inductor cannot change instantaneously, so

$$i_L(0_+) = 0A$$

The general solution for a first order RL circuit is

$$i(t) = \frac{V}{R}\left(1 - e^{\frac{-Rt}{L}}\right)$$

$$i(2) = \frac{50V}{20\Omega}\left(1 - e^{\frac{-(20\Omega)(1.25s)}{8H}}\right)$$

$$= 2.39A$$

The energy in the inductor is found using

$$W_L = \tfrac{1}{2}Li^2$$

$$W_L = \tfrac{1}{2}(8H)(2.39)^2 = 22.85J$$

$$i = C\frac{dv}{dt} + \frac{v}{R}$$

14.11

Applications are an opportunity to demonstrate familiarity, comfort, and comprehension of the topics.