Chapter 14 – Waveforms

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14.1 Introduction

Signals and waveforms are not normally part of the study for electric machines. However, with the growing amount of electronic control and with distortion on the power line from switched mode power supplies, the waveform is often complex. Therefore, this chapter is provided as a reference to assist with those challenges.

Signals that are encountered can be a constant, direct current (DC), they can be repetitive, alternating current (AC), or they can be short term, transients. The circuit elements respond differently to each type signal. This chapter will address waveforms and tools to analyze their impact on systems performance. The time domain signal response or solution contains all the components.

- $y(t) = F + (I - F)e^{-t/\tau}\cos(\omega t + \theta)$

14.2 Waveforms

By far, the sinusoid is the most common repetitive waveform in electrical systems. It is the physical result due to the rotational motion of machines in a magnetic field.

The waveform definitions follow.

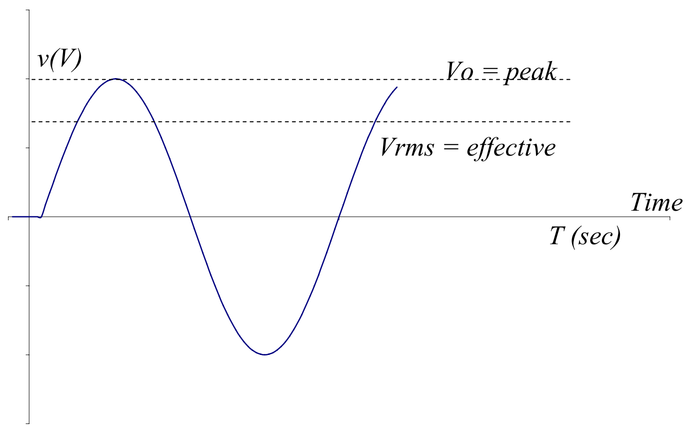
$$voltage = v = V_o \cos \omega t$$

$$frequency = f = \frac{1}{T} \quad (hertz)$$

$$\omega = 2\pi f \qquad (radians/\text{sec})$$

$$average\,value = V_{DC} = \frac{1}{T}\int_0^T dt = 0 \text{ for sinusoid}$$

$$effective\,value = V_{RMS} = \frac{V_o}{\sqrt{2}} = \sqrt{\frac{1}{T}\int_0^T v^2 dt}$$



For multiple waveforms, use superposition. For effective or root mean square (RMS), this is square root of the sum of the squares.

$$V_{RMS} = \sqrt{V_{RMS1}^{\ 2} + V_{RMS2}^{\ 2} + \dots}$$

Generally, AC values of V & I are given in RMS. The frequency is assumed constant.

For a 100Volt, 60Hz voltage waveform

$$v = 100\sqrt{2}\cos(2\pi 60t)$$

$$V_o = 100\sqrt{2}$$

$$V_O = 141 volts$$

14.3 Transients

Transients are waveforms that exist for a short period of time. Waveforms are determined by the circuit elements. Since there are only three elements, the most complex circuit is a second order. The characteristic solution for a systems circuit is the time varying equation that describes the exponential decay after a signal is applied. The variable, *y*, can represent either current or voltage.

$$y(t) = F + (I - F)e^{-t/\tau}\cos(\omega t + \theta)$$

where

F = final value (t=∞)

I = Initial Value (t=0)

τ = time constant

$$\omega = 1/\sqrt{LC}$$

13.3.1 First Order Transients

First order systems are very common, since they are the model of a simple system. First order systems have a resistor and either a capacitor or an inductor.

First Order Circuits

RC or RL

Form:

$$v = L\frac{di}{dt} + Ri$$

$$i = C\frac{dv}{dt} + \frac{v}{R}$$

Characteristic Solution

Response to a step input (DC)

$$y(t) = F + (I - F)e^{-t/\tau}$$

For Capacitor: $i = C\frac{dv}{dt}$

Voltage does not change instantaneously

Open circuit under DC conditions

Capacitor discharges to $V_C(\infty) = 0$

Initial voltage = source voltage

For Inductor: $v = L\frac{di}{dt}$

Current does not change instantaneously

Short circuit under DC conditions

Inductor dissipates to $I_L(\infty) = I$

Initial current = source current

Process:

Find τ

τ = time constant= RC or L/R

= time for exponent to be $e^{-1} \Rightarrow \frac{1}{e}$

use equivalent circuit w/o source to get RC or RL (Thevenin Impedance)

deactivate all the sources and replace with internal Z

reduce to single equivalent RC or RL

Find $y(0)$

use circuit (KVL) w/ element as source

Find y(final)

use circuit (KVL) w/ element as limit

Plot:

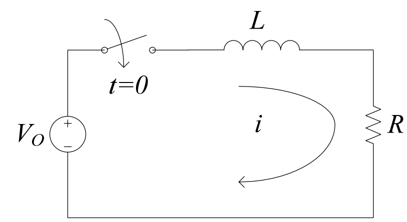
Initial slope = $\frac{F-I}{\tau}$

Transfer function = $\frac{response}{excitation} = \frac{output}{input}$

14.3.2 RL Circuits

Standard calculus form

$$V_o = L\frac{di}{dt} + Ri$$



Inductor is short circuit in final state.

$$i(L)_0 = 0$$

$$i(L)_f = \frac{V_o}{R}$$

$$\tau = \frac{L}{R}$$

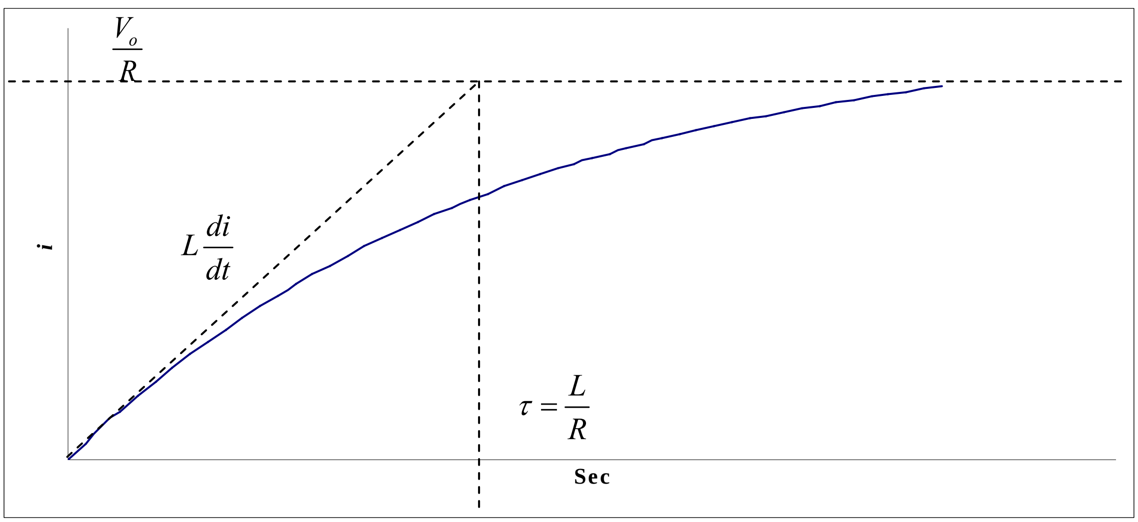
General solution

$$y(t) = F + (I - F)e^{-t/\tau}$$

Current solution

$$i = \frac{V_o}{R} + (0 - \frac{V_o}{R})e^{-\frac{t}{\tau}}$$

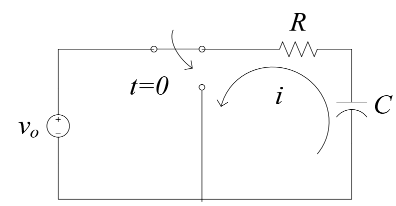
$$i = \frac{V_o}{R}(1 - e^{-\frac{Rt}{L}})$$



14.3.3 RC Circuits

Standard calculus form

$$i = C\frac{dv}{dt} + \frac{v}{R} \text{ (calculus form)}$$



Capacitor is open circuit in final state.

i, v_c cannot change instantaneously

$$v(C)_0 = V_o$$

$$v(C)_f = 0$$

$$\tau = RC$$

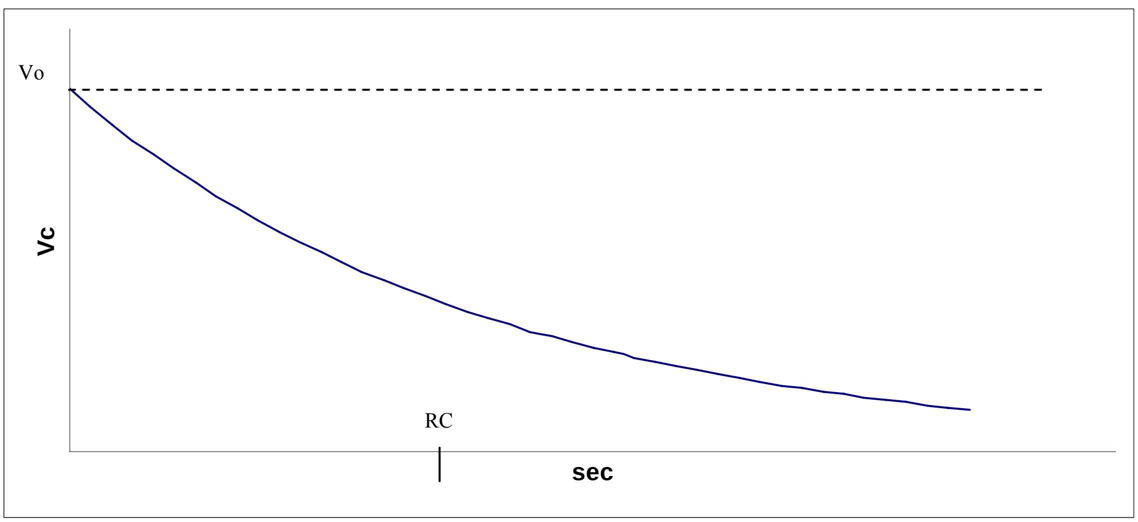
General solution

$$y(t) = F + (I - F)e^{-t/\tau}$$

Voltage solution

$$v = V_F + (V_I - V_F)e^{-t/\tau}$$

$$v_c = V_o e^{-t/\tau}$$

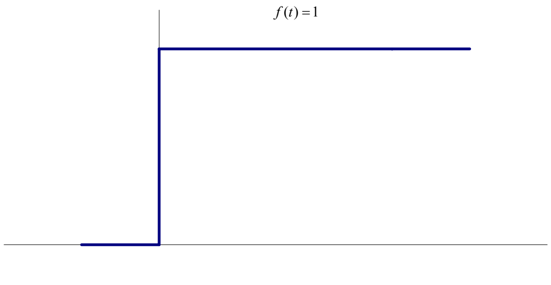
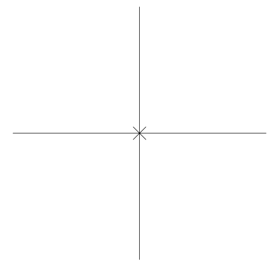
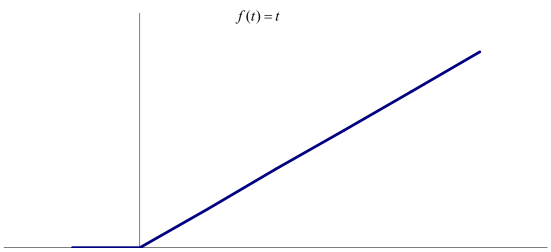
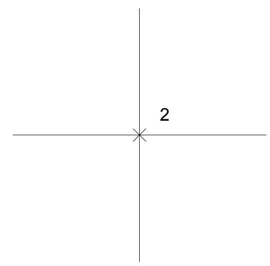
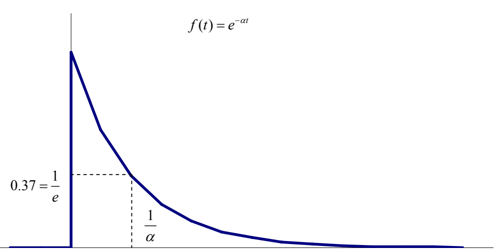
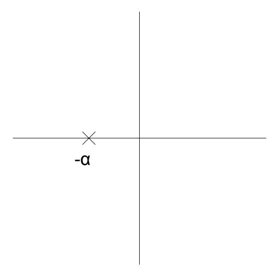
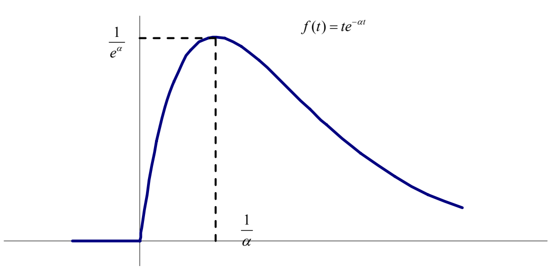
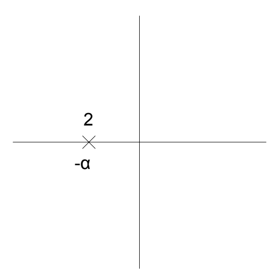


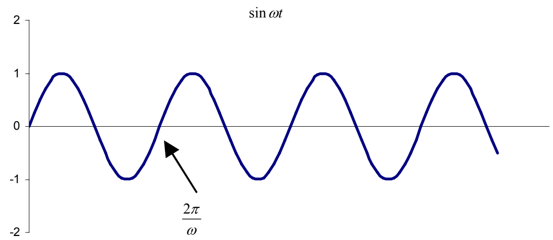
14.4 LaPlace

A standard waveform is defined in terms of time and frequency. A mathematical transform is often used to provide a different mathematical tool. The phasor representation is one transform that applies to steady state alternating circuits. LaPlace transforms are used for many manipulations of the inductor and capacitor elements. The function can be transformed from time to the s domain, which represents a stationary and rotational component.

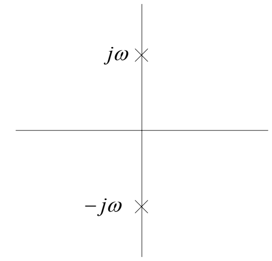
$$s = \sigma + j\omega$$

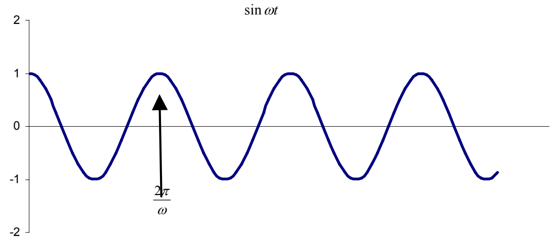
The most used transform pairs are illustrated.

$f(t)$	$f(t)$	$\leftrightarrow$	$F(s)$	$I_m(s)$
	1	$\leftrightarrow$	$\frac{1}{s}$	
	t	$\leftrightarrow$	$\frac{1}{s^2}$	
	$e^{-\alpha t}$	$\leftrightarrow$	$\frac{1}{s+\alpha}$	
	$te^{-\alpha t}$	$\leftrightarrow$	$\frac{1}{(s+\alpha)^2}$	

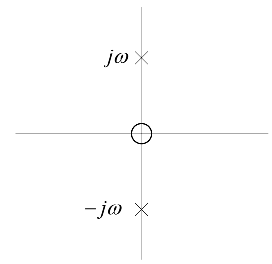


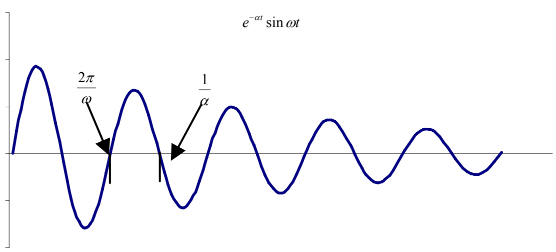
$$\sin \omega t \quad \leftrightarrow \quad \frac{\omega}{s^2 + \omega^2}$$



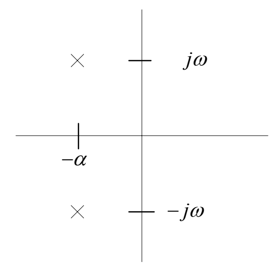


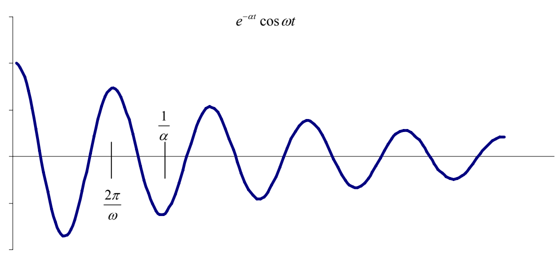
$$\cos \omega t \quad \leftrightarrow \quad \frac{s}{s^2 + \omega^2}$$



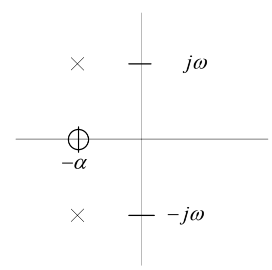


$$e^{-\alpha t} \sin \omega t \quad \leftrightarrow \quad \frac{\omega}{(s+\alpha)^2 + \omega^2}$$





$$e^{-\alpha t} \cos \omega t \quad \leftrightarrow \quad \frac{s+\alpha}{(s+\alpha)^2 + \omega^2}$$



14.5 LaPlace Operational Rules

The mathematical manipulation of the time function and the LaPlace transform follows defined rules.

1) $f_1(t) + f_2(t) \leftrightarrow F_1(s) + F_2(s)$

2) $a\,f(t) \leftrightarrow a\,F(s)$

3) $\dfrac{d\,f(t)}{dt} = f'(t) \leftrightarrow sF(s) - f(0)$

4) $f''(t) \leftrightarrow s^2F(s) - sf(0) - f'(0)$

5) $\int_0^t f(t')dt' \leftrightarrow \dfrac{1}{s}F(s)$

14.5.1 Partial Fraction Expansion

$$F(s) = \frac{N(s)}{D(s)} = \frac{N(s)}{(s-p_1)(s-p_2)\ldots} = \frac{K_1}{(s-p_1)} + \frac{K2}{(s-p_2)} + \ldots$$

$$K_1 = (s-p_1)F(s)\Big|_{s=p_1} = \frac{N(p_1)}{(p_1-p_2)(p_1-p_3)\ldots} jK_2 = \ldots jK_3 = \ldots$$

$$f(t) = K_1 e^{p_1 t} + K_2 e^{p_2 t} + \ldots$$

14.5.2 Alternate Approach

$F_1(s), F_2(s)$ are from the table, known transforms

$$F(s) = K_1F_1(s) + K_2F_2(s) + \ldots$$

Right hand side to common denominator, equate numerators, solve for K_1, K_2…

$$f(t) = K_1f_1(t) + K_2f_2(t) + \ldots$$

Impedance in s-domain

Impedance	$Z = \frac{V}{I}$	$Z_R = R$	$Z_L = sL$	$Z_C = \frac{1}{sC}$
Admittance	$Y = \frac{1}{Z} = \frac{I}{V}$	$Y_R = \frac{1}{R}$	$Y_L = \frac{1}{sL}$	$Y_C = sC$

14.6 Fourier Series

Any alternating waveform can be represented by the summation of a fundamental sine wave and its multiples called harmonics. This summation is called a Fourier series.

$$y = Y_0 + Y_1 \sin(1\omega t + \theta_1) + Y_2 \sin(2\omega t + \theta_2) + \ldots + Y_n \sin(n\omega t + \theta_n)$$

The term y is the instantaneous value at any time. It can be either current or voltage.

The Y_0 term is the constant offset, average, or DC component. The *Y* terms are the maximum amplitude for each of the harmonic frequencies. The angular frequency ω is $2\pi f$. The phase shift angle represents the time delay between the reference voltage waveform and the current. The n subscript and coefficient of frequency indicates the harmonic number.

The time domain is a plot of the *Y* amplitude versus time for the curve. The frequency spectrum is a plot of harmonic amplitude versus harmonic frequency number.

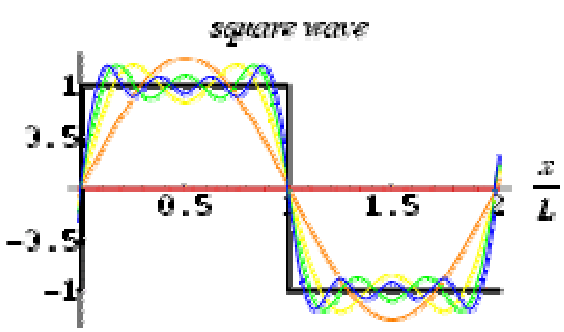
An odd function is created with the sum of the odd harmonics. A sine wave is the basic example. If the waveform has the pattern of a fundamental sine wave, then it is odd.

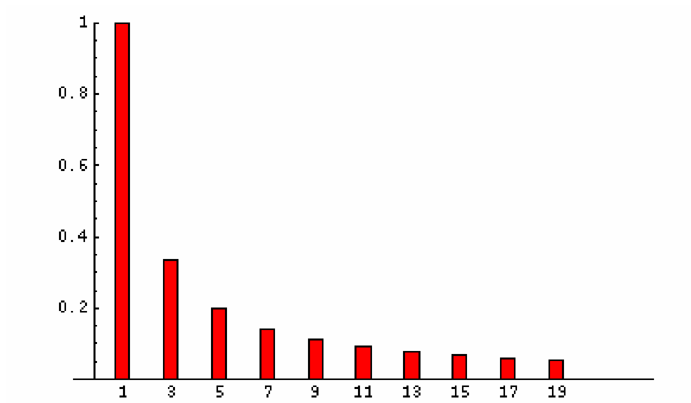
$$y(t) = -y(-t)$$

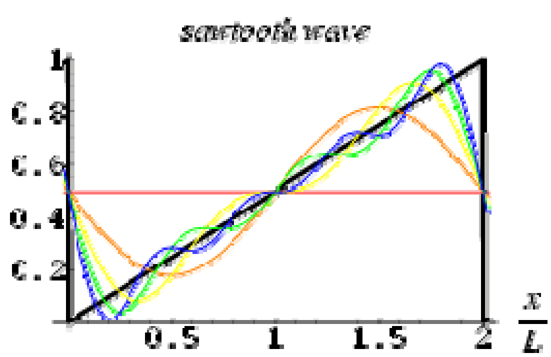
An even function is created with the sum of the even harmonics. A cosine is the basic example. If the waveform has the pattern of a fundamental cosine wave, then it is even.

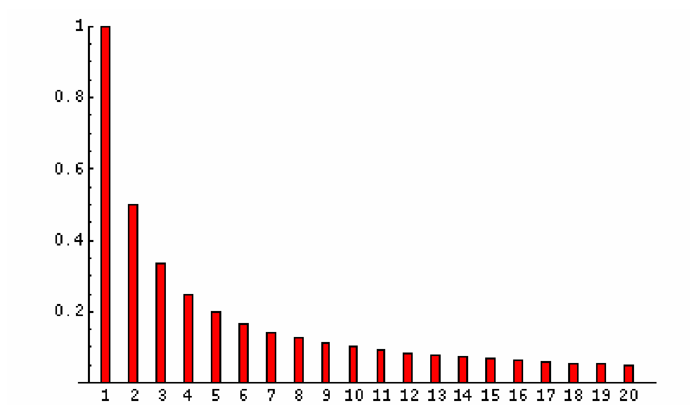
$$y(t) = y(-t)$$

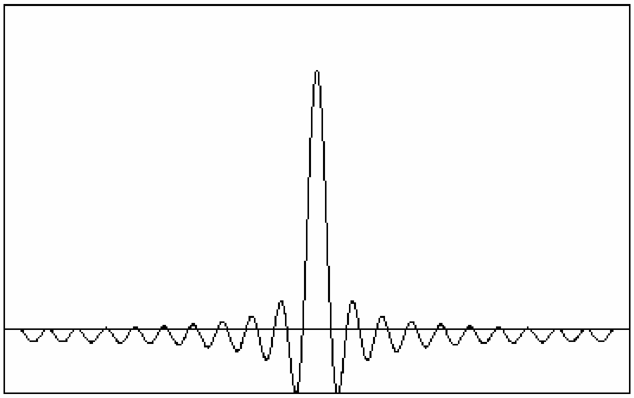
A function that contains both even and odd harmonics will have spikes. A pulse and sawtooth are examples.

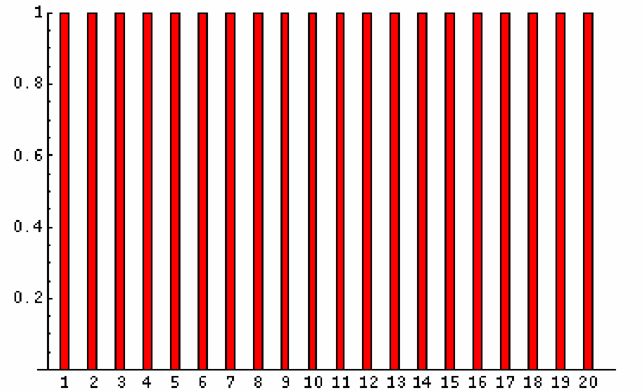


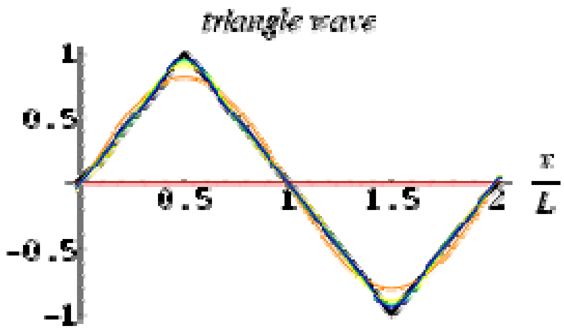


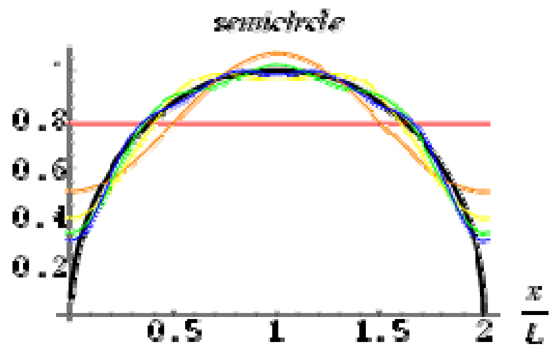












The very definition of Fourier series indicates the series can take several forms. A cosine is an orthogonal shift to a sine wave. As a result, a common representation is to use cosine terms for the even harmonics and sine terms for the odd. Then the even harmonics become odd coefficients for the cosine terms. Although this is a common representation, it is not as easy to visualize or to obtain a spectrum as the simple sinusoidal form.

The Fourier series can be decomposed into the sum of even and odd parts.

$$f(t) = f_e(t) + f_o(t)$$

The even part can be represented by the Fourier series

$$f_e(t) = \frac{a_o}{2} + \sum_{n=1}^{\infty} a_n \cos(n\omega t)$$

The odd part can be represented by the Fourier series

$$f_o(t) = \sum_{n=1}^{\infty} b_n \sin(n\omega t)$$

The coefficients are similar.

$$a_o = \frac{1}{T}\int_0^T f(t)dt$$

$$a_n = \frac{2}{T}\int_0^T f(t)\cos(n\omega_0 t)dt$$

$$b_n = \frac{2}{T}\int_0^T f(t)\sin(n\omega_0 t)dt$$

14.7 Signals - Modulation

Modulation is the process of combining two signals into one combined waveform. The combination can be through an adder or multiplier. The mathematical result looks very much like the Fourier series.

Modulation is similar to wrapping a paper note around a rock and tossing the combination. The carrier wave, or rock, provides a vehicle for passing the information. The information is on the paper note.

In its basic form, the carrier is a single waveform.

$$x(t) = A_c \sin(2\pi f_c t + \phi_c)$$

A_c = amplitude, f = frequency, and θ is the phase shift. Therefore only the amplitude, frequency, or phase can be changed or modulated.

The message, information, or baseband has a similar form. Usually the message wave has a fixed or zero phase shift.

$$m(t) = A_m \sin(2\pi f_m t + \phi_m)$$

14.7.1 Modulation Types

There are numerous variations to the types of modulation.

There are three analog modulation techniques based on the variables in the waveform..

Amplitude modulation (AM)
Frequency modulation (FM)
Phase modulation (PM)

Special variations of these techniques have unique characteristics that affect bandwidth and power.

Angle modulation includes both frequency and phase modulation, since they are operated on by the sinusoid.
Double sideband modulation (DSB) is AM with the carrier removed.
Single-sideband modulation (SSB) is DSB with one of the sidebands removed.

There are three fundamental sampling or digital modulation techniques.

Pulse amplitude modulation (PAM)
Pulse frequency modulation (PFM)
Pulse phase modulation (PPM)

Variations of these techniques result in a variety of keying processes. The process of modulation and demodulation is called a modem.

Pulse code modulation includes both frequency and phase.
Amplitude shift key modulation (ASK)
Frequency shift key modulation (FSK)
Binary-phase shift key modulation (BPSK)
Quadrature-phase shift key modulation (QPSK)
Quadrature amplitude modulation (QAM)

14.7.2 Amplitude Modulation (AM)

Amplitude modulation mixes the information or message with the carrier amplitude. The general form of amplitude modulation is to add a function of the message to the carrier amplitude.

$$y(t) = [A_c + k_a m(t)]\sin(2\pi f_c t)$$

For a single waveform, k_a is unity.

$$k_a = 1$$

The amplitude varies with the carrier and the signal. The expanded form illustrates the three components, carrier + lower sideband - upper sideband.

$$y(t) = [A_c + m(t)]\sin(2\pi f_c t)$$

$$y(t) = [A_c + A_m \sin(2\pi f_c t]\sin(2\pi f_c t)$$

$$y(t) = A_c \sin(2\pi f_c t) + \frac{1}{2} A_m \cos(2\pi f_c - 2\pi f_m)t - \frac{1}{2} A_m \cos(2\pi f_c + 2\pi f_m)t$$

The modulation index is the depth of the variation around the original level of the carrier, A_c. When multiplied by 100, it is the percent modulation.

$$m_{am} = \frac{\Delta A}{A_c} = \frac{A_m}{A_c}$$

The power in an AM signal is the sum of the power in the carrier and the power in the signal.

$$P = P_c + P_m$$

$$= (\frac{A_c^2}{2})((km(t))_{ave}^2 + 1)$$

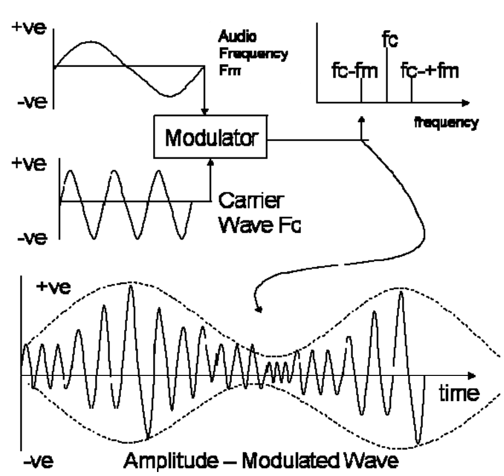
AM signals can be demodulated with an envelope detector or a synchronous demodulator.

A double sideband (DSB) signal would contain the upper and lower sideband information but would not have the carrier. DSB signals can be demodulated with a synchronous demodulator. A Costas loop is a common technique.

Single sideband (SSB) can be either the lower or upper sideband information only without the carrier or the other sideband. AM signals can be demodulated with a synchronous demodulator or by carrier reinsertion and envelope detector.

The bandwidth has a lower frequency of f_c-f_m, center frequency f_c, and an upper frequency of f_c+f_m.

$$BW = f_h - f_l$$



14.7.3 Angle Modulation

Angle modulation mixes the signal as a component of the carrier sinusoid which includes the frequency and phase terms. In essence the signal becomes the phase term.

$$x(t) = A_c \sin(2\pi f_c t + \phi_c)$$

$$y(t) = A_c \sin[2\pi f_c t + m(t)]$$

$$y(t) = A_c \sin(2\pi f_c t)\cos(m(t)) + A_c \cos(2\pi f_c t)\sin(m(t))]$$

$$y(t) = A_c \sin(2\pi f_c t)\cos(A_m \sin(2\pi f_m t)) + A_c \cos(2\pi f_c t)\sin(A_m \sin(2\pi f_m t))]$$

This is obviously a very complex function with numerous frequency components. There are infinite sidebands to the signal. However, the amplitude of most deteriorates quickly. Frequency modulation and phase modulation each use select components of this waveform.

The phase deviation or shift is a function of the message or information. As discussed earlier, it is assumed that the message phase shift is zero. The function, k_p, is the phase modulation index.

$$\phi(t) = k_p m(t)$$

The instantaneous phase is the carrier angle added to the signal. This is the angle within the carrier wave sin term.

$$\phi_i(t) = 2\pi f_c t + \phi(t)$$

The instantaneous frequency is the change of instantaneous phase with time. The instantaneous frequency is the carrier frequency plus the frequency deviation.

$$\begin{aligned} f_i &= \frac{d}{dt}\phi_i \\ &= f_c + \frac{d}{dt}\phi(t) \\ &= f_c + \Delta f \end{aligned} \qquad \begin{aligned} \omega_i &= \frac{d}{dt}\phi_i \\ &= 2\pi f_c + \frac{d}{dt}\phi(t) \\ &= \omega_c + \Delta\omega \end{aligned}$$

The frequency deviation is the change in the phase, which is the change in the message with time.

$$\Delta\omega = \frac{d}{dt}\phi(t) = \frac{d}{dt}km(t)$$

The message bandwidth is the frequency of modulation, f_m.

$$BW_m = f_m$$

The bandwidth of an FM & PM signal is approximated using Carson's rule.

$$\begin{aligned} BW_y &= 2(\Delta f + f_m) \\ &= 2(m_{fm} + 1) f_m \end{aligned}$$

14.7.4 Frequency Modulation (FM)

Frequency modulation mixes the information or message with the carrier frequency. The amplitude is constant. The result is the carrier varies above and below its idle or normal frequency, f_c. As the voltage amplitude of the modulating signal increases in the positive direction from A to B, the frequency of the carrier is increased in proportion to the modulating voltage.

Frequency modulation is adding the carrier frequency and a function of the message.

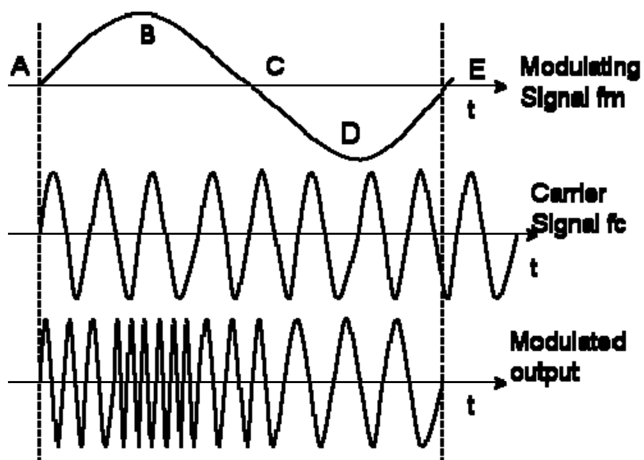
$$f_i(t) = f_c + k_f m(t)$$

The modulation index or factor is the maximum deviation in frequency, Δf, divided by the modulation frequency. When multiplied by 100, it is the percent modulation.

$$m_{fm} = \frac{\Delta f}{f_m}$$

The frequency modulator constant is the frequency deviation divided by the amplitude of the modulating or message signal.

$$k_f = \frac{\Delta f}{A_m}$$



14.7.5 Phase Modulation

Phase modulation is another component of angular modulation that is sometimes referred to indirect FM. Note that the phase is part of the sinusoid. Here, the amount of the carrier frequency shift is proportional to both the amplitude and frequency of the modulating signal. The phase of the carrier is changed by the change in amplitude of the modulating signal. The modulated carrier wave is lagging the carrier wave when the modulating frequency is positive.

Phase modulation is manipulation of the angle of the carrier and a function of the signal.

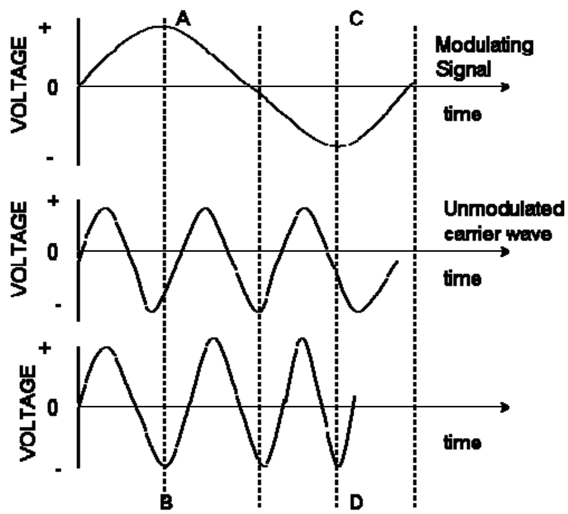
$$\phi_i(t) = 2\pi f_c t + k_p m(t)$$

The modulation index is the peak phase variation.

$$m_{pm} = \Delta\phi(t) = \frac{\Delta f}{f_m}$$

The phase modulation constant depends on both the frequency and amplitude. It is the ratio of the phase deviation to the message.

$$k_p = \frac{\phi(t)}{m(t)}$$



Phase Modulated carrier wave

14.7.6 Sampled Messages

A message, *m(t)* can be recreated from uniformly spaced samples. The sampling frequency, called the Nyquist frequency f_N,, must be at least twice as fast as the highest frequency being recreated.

$$f_N = \frac{1}{T_s} = 2f$$

14.7.7 Digital - Pulse Modulation

Pulse or digital modulation is frequently used to transmit sampled messages. Analog to digital conversion is a two step process. First, sampling changes the analog source to a series of discrete values, called sample. Second, quantization, converts each sample to a number. The number of quantization levels, q, is the two power of the number of bits.

$$q = 2^n$$

The bandwidth required is inversely proportional to the inverse of twice the pulse length or duration, T. This is called the Shannon bandwidth when the Dimensionality, D is included. For minimum bandwidth, D=1.

$$BW_S = \frac{D}{2T}$$

The message bandwidth, W, and the number of bits determine the minimum modulated bandwidth, BW.

$$BW \propto nW = 2W \log_2 q$$

14.8 Signal transmission

14.8.1 dBm

Signal power can be measured in watts. However, comparison values and small signals are measured in decibels.

$$db = 10\log_{10}(\frac{P_{signal}}{P_{ref}})$$

When the reference is on milliwatt, then the decibels are reference as dBm.

$$dBm = 10\log_{10}(\frac{P_{signal}}{1mW})$$

As a result a 1 milliwatt signal is 0dbM.

$$\therefore 0\ dBm = 1\ mW$$

14.8.2 Noise

Noise is a random or background signal that may interfere with the message or information. Signal-to-noise ratio is an indication of the power ratio between the desired information and the background noise. The symbols are SNR or S/N.

$$S/N = \frac{P_{signal}}{P_{noise}}$$

Often the expression is in terms of decibels (dB).

$$S/N(db) = 10\log_{10}(\frac{P_{signal}}{P_{noise}}) = 20\log_{10}(\frac{A_{signal}}{A_{noise}})$$

In a digital signal, the number of bits in each value determines the SNR. Noise in a digital signal is dependent on the conversion process. The dynamic range is an expression of the SNR.

$$S/N(db) = DR(db) = 20\log_{10}(2^n)$$

White noise creates a thermal noise power, *P*, in watts that is dependent on the bandwidth, Δf in Hertz and temperature, *T* in degrees Kelvin. This is also the thermal noise that will be created by electron activity in a resistor and is called Johnson noise.

$$P_T = K_B T_K \Delta f$$

$$K_B = 1.3806503 \times 10^{-23} \frac{J}{°K} = \text{Boltzmann's constant}$$

$$T_K = T_C + 273.15^O$$

For current or voltage across the resistor the power has the standard relationships.

$$P = \frac{V^2}{R} = I^2 R$$

Thermal noise at room temperature is dependent on the bandwidth. The units are decibels.

$$P(dB) = -174 + 10\log(\Delta f)$$

The total noise figure for a series of transfer functions or amplifiers is based on the ratio of the noise figure for each stage, *F*, to the gain ratio of each stage, *G*. The noise figure and gain must be converted to the power ratio from dB.

$$F_T = \frac{F_1}{1} + \frac{F_2 - 1}{G_1} + \frac{F_3 - 1}{G_1 G_2} + \ldots$$

14.8.3 Propagation - Transmit

The velocity of propagation of a wave is the distance the wave will travel in one time period. If the distance is one wavelength, λ, then the velocity is the ratio of the wavelength to the frequency.

$$u_p = \frac{d}{t} = \frac{\lambda}{f}$$

In free space, the propagation velocity is the speed of light.

$$c = 2.99\ 792\ 458 \text{ x } 10^8$$

The velocity of a wave on a transmission line is simply the ratio of the distance to the time it takes for the wave to propagate. For a reflected wave, the distance is twice the length because of the trip length and back.

$$u_p = \frac{d}{t}$$

Transmission of waves involves the power density in Watts per square meter. It is the ratio of the power transmitted to the orthogonal area that the waveform strikes. A spherical shape is the normal pattern of an omni-directional wave.

$$P_{\text{density}} = \frac{P_{\text{transmitted}}}{A} = \frac{P_x}{4\pi R^2}$$

$$R = \text{range from antenna, radius of sphere}$$

Antennas can direct power in specific directions. The gain of the antenna is the radiation intensity in a particular direction divided by the power that would be radiated from an omni-directional or isotropic antenna.

$$G = \frac{\text{Effective radiated power}}{\text{Isotropic rated power}}$$

Power is dissipated as a waveform propagates. The attenuation or loss in free space depends on the velocity of light. In other mediums, the velocity of propagation should be used. The loss is dB, distance is m, and frequency is Hz.

$$P_{fs} = 20\log\left(\frac{4\pi d}{c/f}\right)$$

Characteristic impedance is the opposition in a circuit that connected to the output terminals of a line will cause the line to appear infinitely long. It is the electric and magnetic property of the material that impacts the velocity of propagation.

$$Z_0 = \sqrt{\frac{\mu}{\varepsilon}} = \frac{1}{u_p \varepsilon} = u_p \mu$$

The electric property is permittivity in Farads per meter, Fd/m. It is a factor of the free air .

$$\varepsilon = \varepsilon_r \varepsilon_0$$

$$\varepsilon_0 = 8.854\text{x}10^{-12} \text{ Fd/m}$$

The magnetic property is permeability in Henries per meter, Hy/m.

$$\mu = \mu_r \mu_0$$

$$\mu_0 = 4\pi\text{x}10^{-7} \text{ Hy/m}$$

From these three preceding concepts, the impedance of free space air is calculated.

$$Z_0 = 377\Omega$$

Because of the definitions of inductance and capacitance in relation to permeability and permittivity, characteristic impedance can be found in terms of circuit elements.

$$Z_0 = \sqrt{\frac{L}{C}}$$

14.8.4 Reflections

Maximum power transfer occurs when the load is equal to the source or characteristic impedance. When a discontinuity occurs on a line or a load is connected that does not match the characteristic impedance, the waveform will be reflected and oppose the message signal. The reflection coefficient describes both the magnitude and phase shift of the reflection. The coefficient is the ratio of the complex forward voltage to the complex reverse wave voltage.

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{V_f}{V_r}$$

Standing wave ratio is the maximum power over the minimum power transferred. SWR is dependent on the reflection coefficient.

$$SWR = \frac{1+\Gamma}{1-\Gamma}$$

Voltage SWR is the maximum voltage over the minimum voltage nodes. VSWR only contains the magnitude of reflection coefficient.

$$VSWR = \frac{V_{\max}}{V_{\min}} = \frac{1+|\Gamma|}{1-|\Gamma|}$$

The reflection coefficient has the following range of values.

$\Gamma = -1$: maximum negative reflection, line is short-circuited,
$\Gamma = 0$: no reflection, when the line is perfectly matched,
$\Gamma = +1$: maximum positive reflection, line is open-circuited.

At the maximum nodes the waves interfere positively and add. At the minimum nodes, the waves are colliding and subtract.

$$V_{\max} = V_f + V_r = V_f(1+|\Gamma|)$$
$$V_{\min} = V_f - V_r = V_f(1-|\Gamma|)$$

Transmission line properties are defined in terms of propagation constant. Propagation constant is inversely proportional to the wavelength. The distance is measured from the load.

$$\beta = \frac{2\pi}{\lambda}$$

$$V(d) = V^+ e^{j\beta d} + V^- e^{j\beta d}$$

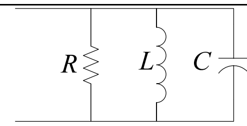
$$I(d) = I^+ e^{j\beta d} + I^- e^{j\beta d}$$

$$Z_{in} = Z_0 \frac{Z_L + jZ_0 \tan(\beta d)}{Z_0 + jZ_L \tan(\beta d)}$$

14.9 RLC System Response

14.9.1 RLC Equations

The three elements, RLC can be arranged in series or its dual parallel. This is a second order system. The analysis of the circuit can be made in many domains. Typically the time domain is the starting point. However, the Calculus required makes the mathematic interpretation difficult. For that reason numerous transforms are used. The math of the transforms will not be developed, but the correspondence is apparent from the table. The duality of the circuits is intriguing.

Function	Series	Parallel
Reference	Same current through all elements	Same voltage across all elements
Diagram	**Error! Not a valid link.**	
Fundamental	$v(t) = L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q$	$i(t) = C\frac{d^2\phi}{dt^2} + \frac{1}{R}\frac{d\phi}{dt} + \frac{1}{L}\phi$
Time	$v(t) = L\frac{di}{dt} + Ri + \frac{1}{C}\int idt$	$i(t) = C\frac{dv}{dt} + Rv + \frac{1}{C}\int vdt$
LaPlace	$V(s) = (Ls + R + \frac{1}{Cs})I(s)$	$I(s) = (Cs + \frac{1}{R} + \frac{1}{Ls})V(s)$
Sinusoidal Steady State	$V(j\omega) = (j\omega L + R + \frac{1}{j\omega C})I(j\omega)$	$I(j\omega) = (j\omega C + \frac{1}{R} + \frac{1}{j\omega L})V(j\omega)$

Several observations can be made about the relationships.

$\frac{d}{dt} = s = j\omega$	$\int dt = \frac{1}{j\omega} = \frac{1}{s}$
$\frac{dq}{dt} = q' = i$	$\frac{d\phi}{dt} = \phi' = v$

14.9.2 System Response

The system response is the solution to the second order equation.

$$y(t) = F + (I - F)e^{-t/\tau}\cos(\omega t + \theta)$$

Time constant is the time it takes for a signal to settle so that the exponential decay.

$$\tau = RC = \frac{L}{R} = \text{time constant}$$

14.9.3 Characteristic Transfer

Transfer functions are often used as a model for a system.

Function	Series	Parallel
Transfer function	$Y(s)=\frac{I(s)}{V(s)}$	$X(s)=\frac{V(s)}{I(s)}$
Characteristic	$Y(s)=\frac{1}{Ls+R+\frac{1}{Cs}}$	$Z(s)=\frac{1}{Cs+\frac{1}{R}+\frac{1}{Ls}}$
Standard form	$Y(s)=\frac{s/L}{s^2+\frac{R}{L}s+\frac{1}{LC}}$	$Z(s)=\frac{s/C}{s^2+\frac{s}{RC}+\frac{1}{LC}}$
Resonance	$Y(s)=\frac{s/L}{s^2+\Delta\omega s+\omega_0^2}$	$Z(s)=\frac{s/C}{s^2+\Delta\omega s+\omega_0^2}$

14.9.4 Resonance

Frequency is inversely related to time. Angular frequency is one complete revolution of cycle of the frequency.

$$\omega = 2\pi f$$

Resonance is a very significant concept that may be a boon or ban to electrical systems. Resonance is the frequency where the magnetic (or inductor) energy equals the electric (or capacitor) energy.

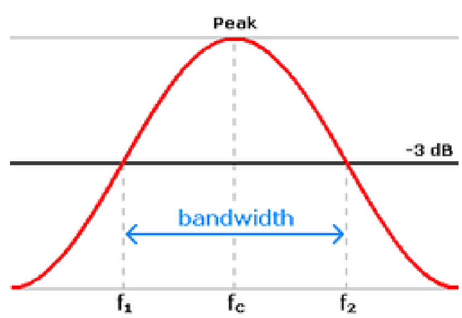
$$\omega_0 = \frac{1}{\sqrt{LC}}$$

Since the energies are balanced, it flows from one to the other resulting in a sinusoidal frequency. The natural frequency is the oscillation determined by the physical properties. Resonant frequency is a created oscillation that matches the natural frequency. Resonance is the frequency at which the input impedance is purely real or resistive.

The frequency response has a roll-off on either side. The transition is called the cut-off frequency.

$$\omega_0^2 = \omega_{cL}\omega_{cH}$$

Bandwidth, $\Delta\omega$, is the range between the upper and lower cut-off frequencies. The bandwidth is also called the pass band or bandpass.



$$\Delta\omega = \omega_{cH} - \omega_{cL}$$

$$\omega_{cL} = \omega_0 - \frac{\Delta\omega}{2}$$

$$\omega_{cH} = \omega_0 + \frac{\Delta\omega}{2}$$

Quality factor or selectivity is the sharpness of the peak at resonance.

$$Q = \frac{\omega_0}{\Delta\omega}$$

Damping is the effect of resistance on the rate that a signal is stabilized to steady state. Undamped implies that there is no resistance, R=0. The *damping coefficient* is dependent on the natural frequency and is inversely proportional to twice the quality factor. Some authors use the symbol alpha, α, rather than zeta, ς. Note this is also the real term of the LaPlace, σ.

$$\zeta = \frac{\omega_0}{2Q} = \frac{R}{2\sqrt{L/C}} = \frac{\text{actual damping}}{\text{critical damping}}$$

The range of values for the camping coefficient reflects how quickly the waveform will settle and whether it will overshoot. Under-damping results in oscillations or ringing, over-damping results in a slow exponential approach to stability, critical-damping is the transition between oscillations and exponential.

$$\zeta < 1 \rightarrow \text{ under-damped = oscillation}$$
$$\zeta = 1 \rightarrow \text{ critical-damped = transition}$$
$$\zeta > 1 \rightarrow \text{ overdamped = exponential}$$

The relationship between the various factors can be described in terms of the quality factor.

$$Q = \frac{\omega_0}{\Delta\omega} = \frac{\omega_0}{2\zeta}$$

Damped resonance, ω_d, is a shift from the resonant frequency caused by the damping.

$$\omega_d^2 = \omega_0^2 - \zeta^2$$

The root of the characteristic equation has the real part as damping coefficient and the imaginary part as the damped resonance. For the second order, there are two roots.

$$s_{1,2} = -\zeta \pm j\omega_d$$

14.9.5 Series Parallel Duality

Comparison of the standard form and the resonance equation reveal the duality of impedance and admittance. The symmetry of the duality resolves to a reciprocal form at resonance.

Function	Series	Parallel
Quality factor	$Q = \frac{X}{R}$	$Q = \frac{R}{X}$
Quality factor	$Q = \frac{1}{R}\sqrt{\frac{L}{C}}$	$Q = R\sqrt{\frac{C}{L}}$

14.9.6 First Order

A first order system has a resistor and either a capacitor or inductor. Therefore, there is no oscillation. However, there is still a cut-off frequency that is the inverse of the time constant.

$$R(j\omega) + \frac{1}{C} = 0 \Rightarrow \omega_c = \frac{1}{RC}$$ Time Constant = RC

$$L(j\omega)^2 + R(j\omega) = 0 \Rightarrow \omega_c = \frac{R}{L}$$ Time Constant = $\frac{L}{R}$

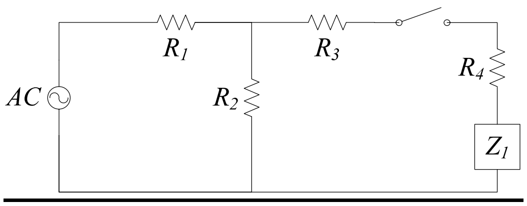
End of chapter

14.10 Exemplars

An exemplar is typical or representative of a system. These examples are representative of real world situations.

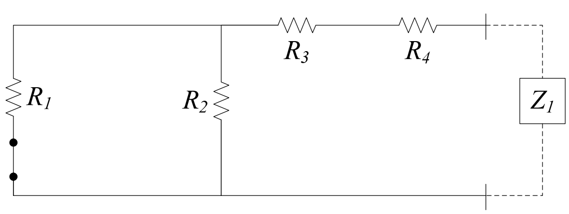
Problem 1

Consider the circuit shown below. R_1 and R_2 are 5Ω resistors. R_3 is a 10Ω resistor and R_4 is a 15Ω resistor. Z_1 is a 20μF capacitor, and V_1 is a 120V source. The time constant of the circuit is most nearly



(A) 85 μS
(B) 138 μS
(C) 550 μS
(D) 400 μS

SOLUTION:
Redraw the circuit to make it easier to see



The resistances can be combined to determine the equivalent resistance of the circuit.

$$R_{eq} = R_4 + R_3 + (R_1 // R_2)$$
$$R_{eq} = 15\Omega + 10\Omega + (5\Omega // 5\Omega)$$
$$R_{eq} = 25\Omega + 2.5\Omega = 27.5\Omega$$

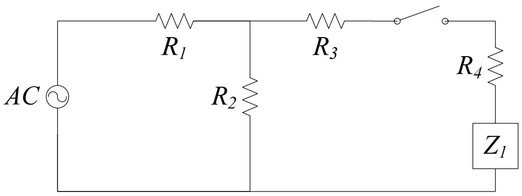
The time constant of a RC circuit is

$$\tau = R_{eq}C_{eq} = (27.5\Omega)(20\mu F) = 550\mu S$$

The answer is (C)

Problem 2

Consider the circuit shown in the problem above, and recreated below. R1 and R2 are 15Ω resistors. R3 is a 20Ω resistor and R4 is a 15Ω resistor. Z1 is a 20mH capacitor, and V1 is a 120V, 60Hz source. The switch has been closed for a significant period of time. The voltage across the inductor is most nearly.



(A) $25.4\angle 55°$

(B) $10.5\angle 80°$

(C) $50.7\angle -60°$

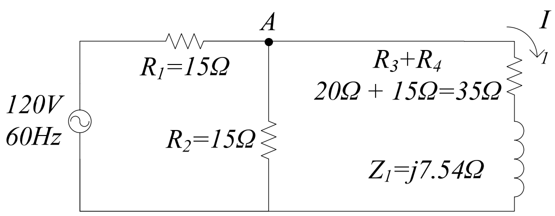
(D) $61.8\angle 90°$

SOLUTION

Impedance of the Inductor Z_1

$$Z_1 = j2\pi 60(20mH) = j7.54\Omega$$

Redraw with all impedances



$$R_A = R_3 + R_4 + Z_1 = 35 + j7.54\Omega$$

$$R_B = 15\Omega // R_A = 10.6 + j0.664\Omega$$

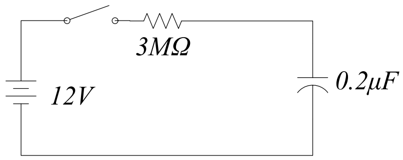
$$V_A = \frac{120(R_B)}{(R_1 + R_B)} = \frac{120V(10.6 + j0.664\Omega)}{(25.6 + j0.664\Omega)} = 49.73 + j1.822$$

$$V_{Z_1} = \frac{V_A(Z_1)}{(R_A + Z_1)} = \frac{(49.73 + j1.822V)(j7.54\Omega)}{(35 + j7.54\Omega)} = 1.83 + j10.31V = 10.5\angle 79.94V$$

The answer is (B)

Problem 3

What is the time constant of the figure shown?



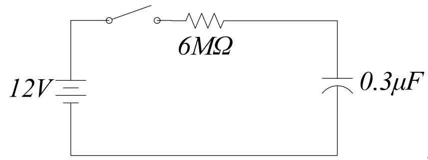
SOLUTION:

The time constant of an RC circuit is

$$\begin{aligned}\tau &= RC \\ &= \left(3\times10^{6}\right)\left(0.2\times10^{-6}\right) \\ &= 0.6 \text{ seconds}\end{aligned}$$

Problem 4

In the figure below, the switch has been open for a significant period of time and is closed at t=0. What is the current in the capacitor at $t=0_+$?



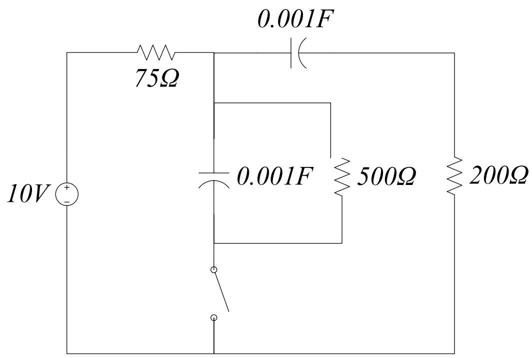
SOLUTION:

The capacitor, at t=0+, acts as a short circuit. The current through the capacitor then is determined by the voltage and the resistance

$$i_c\Big|_{t=0+} = \frac{V}{Z} = \frac{12V}{6\times10^6\Omega} = 2\times10^{-6}\,A$$

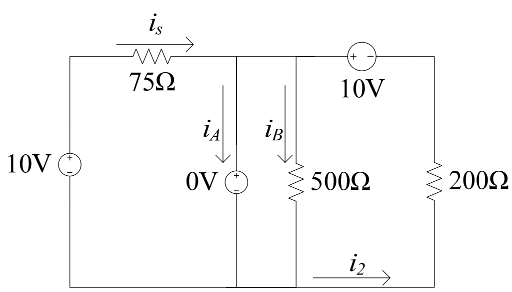
Problem 5

In the figure below, the switch has been open for a significant period of time, and is then closed at t=0. What is the current through the two capacitors at $t=0_+$?



SOLUTION:

If the switch is opened for a significant period of time the capacitor on top of the circuit is charged to 10V, and the capacitor in the middle of the circuit is discharged to 0V. At $t=0_+$, the capacitors are modeled as voltage sources with the charged voltages. The equivalent circuit is shown below



The voltage across the 500Ω resistor is 0V, so i_B=0A.

KVL on the left loop is

$$10V - i_s(75\Omega) - 0V = 0$$

$$\Rightarrow i_s = \frac{10V}{75\Omega} = 0.1\overline{33}A$$

KVL on the right loop is

$$10V + 0V + i_2(200\Omega) = 0$$

$$\Rightarrow i_2 = \frac{10V}{200\Omega} = 0.05A$$

KCL

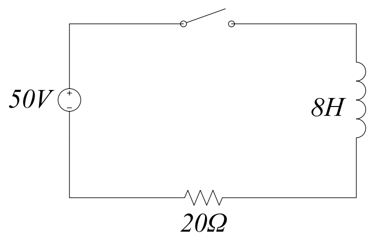
$$i_s - i_A - i_B - i_2 = 0$$
$$0.133A - i_A - 0 - 0.05A = 0$$
$$i_A = 0.133 - 0.05 = 0.083A$$

The current through the top capacitor is i_2=0.05A

The current through the middle capacitor is i_A = 0.083A

Problem 6

In the figure below, the switch has been open for a significant period of time. The switch is closed at t=0. Find the current through the resistor at t=0_+, and at t=1.25s. Find the energy in the inductor at t=1.25s.



SOLUTION:

The current in an inductor cannot change instantaneously, so

$$i_L(0_+) = 0A$$

The general solution for a first order RL circuit is

$$i(t) = \frac{V}{R}\left(1 - e^{\frac{-Rt}{L}}\right)$$

$$i(2) = \frac{50V}{20\Omega}\left(1 - e^{\frac{-(20\Omega)(1.25s)}{8H}}\right)$$

$$= 2.39A$$

The energy in the inductor is found using

$$W_L = \tfrac{1}{2}Li^2$$

$$W_L = \tfrac{1}{2}(8H)(2.39)^2 = 22.85J$$

Problem 7

A carrier wave of 12 MHz is amplitude modulated by an audio signal of 1.5 kHz. What are the upper and lower limits of the resulting modulated signals bandwidth?

SOLUTION:

$$f_c = 12 \times 10^6 \, Hz$$

$$f_M = 1.5 \times 10^3 \, Hz$$

Lower sideband frequency - $f_C - f_M = 12 \times 10^6 - 1.5 \times 10^3 \, Hz = 11,998,500 Hz$

Upper sideband frequency - $f_C - f_M = 12 \times 10^6 + 1.5 \times 10^3 \, Hz = 12,001,500 Hz$

Problem 8

A 110 MHz carrier is frequency modulated by a 65kHz information signal. The information signal has a 1V amplitude, and a frequency modulator constant of 100Hz/V. What is the bandwidth?

SOLUTION:

Carson's Rule

$$BW \approx 2(ak_f + f_m)$$

$$BW = 2\left((1V)\left(100\frac{Hz}{V}\right) + 65\times10^3 Hz\right)$$

$$= 130,200 Hz \approx 130 kHz$$

14.11 Applications

Applications are an opportunity to demonstrate familiarity, comfort, and comprehension of the topics.